Introduction To Dynamical Systems And Chaos

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These notes are primarily based on the textbook by Strogatz [1].

1 One-Dimensional Flows

1.1 Definition of Dynamical Systems

Definition 1.1 (Dynamical system). A *dynamical system* is a set of rules that governs the time evolution of a set of *state variables*.

There are two main types of dynamical systems: (1) differential equations, (2) iterated maps.

Dynamical systems are classified similarly to ODEs. A dynamical system is said to be *autonomous* if the right-hand side does not depend explicitly on time t. A dynamical system is said to be *linear* if the function $f(x)$ is linear in x.

Definition 1.2 (State variables). State variables are a collection of variables of interest that describe the mathematical state of a dynamical system.

Definition 1.3 (One-dimensional (first-order) dynamical system). The general first-order (or one-dimensional) dynamical system is

$$\dot{x} = f(x),$$

where $x(t)$ is a real-valued function of time t, and $f(x)$ is a smooth real-valued function of x.

Definition 1.4 (Nonautonomous equations). Time-dependent (or nonautonomous) equations of the form

$$\dot{x} = f(x, t)$$

should be regarded as a two-dimensional or second-order system.

1.2 Fixed Point and Stability

Definition 1.5 (Equilibrium solution / fixed point). Equilibrium solutions (or *fixed points*) are points x^* in the phase space for which the dynamical system does not change in time, i.e.,

$$\dot{x} = 0.$$

Fixed points correspond to stagnation points of the flow. They are also referred to as steady/constant/rest solutions.

Definition 1.6 (Phase space / state space). The *phase* (or *state space*) is the space in which all possible states of a dynamical system are represented. For one-dimensional ODEs, the phase space is the set of all real numbers $\mathbb{R}$.

Definition 1.7 (Stability of fixed points). Let x^* be a fixed point.

(1) x^* is *stable* if a solution that starts near it stays near it.
(2) x^* is *asymptotically stable* if it is stable and, moreover, the solution approaches it as $t \to \infty$.
(3) x^* is *unstable* if a sol. that starts near it moves away from it.
(4) x^* is *semi-stable (half-stable)* if it attracts from one side and repels from the other.
(5) x^* is *globally stable* if it is approached from all initial conditions.

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Definition 1.8 (Phase portrait). A *phase portrait* is a drawing of the trajectories of a dynamical system in the phase space.

Definition 1.9 (Trajectory). A *trajectory* represents the solution $x(t)$ of the ODE starting from the initial condition x_0.

1.3 Linear stability analysis

Consider the autonomous ODE

$$\dot{x} = f(x),$$

and let x^* be a fixed point, that is,

$$f(x^*) = 0.$$

Let $\varepsilon(t)$ be a small perturbation with $|\varepsilon(t)| \ll 1$ for all t, and write

$$x(t) = x^* + \varepsilon(t).$$

Using Taylor expansion of f around x^*,

$$f(x) = f(x^*) + \varepsilon(t) f'(x^*) + \frac{\varepsilon(t)^2}{2} f''(x^*) + \cdots .$$

From $\dot{x} = f(x)$, x^* is constant, and using $f(x^*) = 0$,

$$\dot{\varepsilon}(t) = \varepsilon(t) f'(x^*) + \frac{\varepsilon(t)^2}{2} f''(x^*) + \cdots .$$

Since $|\varepsilon| \ll 1$, the higher-order terms are negligible, and we obtain the linearized equation

$$\dot{\varepsilon}(t) = f'(x^*)\, \varepsilon(t).$$

The solution is

$$\varepsilon(t) = \varepsilon(0) e^{f'(x^*) t}.$$

- If $f'(x^*) > 0$, then $\varepsilon(t) \to \infty$ as $t \to \infty$. The perturbation grows exponentially, and the fixed point x^* is unstable.
- If $f'(x^*) < 0$, then $\varepsilon(t) \to 0$ as $t \to \infty$. The perturbation decays exponentially, and the fixed point x^* is stable.
- If $f'(x^*) = 0$, the linearization test is inconclusive, and higher-order terms must be considered.

THEOREM 1.10 (EXISTENCE AND UNIQUENESS THEOREM). *Consider the initial value problem*

$$\dot{x} = f(x), \quad x(0) = x_0.$$

Suppose that $f(x)$ and $f'(x)$ are continuous on an open interval R of the x-axis, and suppose that $x_0 \in R$. Then the problem has a solution $x(t)$ on some time interval $(-\tau, \tau)$, and the solution is unique.

Trajectories for a vector field on the real line are forced to increase or decrease monotonically or remain constant. The phase point never reverses direction. There are no periodic solutions to $\dot{x} = f(x)$ on the real line.

These results correspond to the fact that $\dot{x} = f(x)$ corresponds to flow on a line. If you flow monotonically on a line, you will never come back to your starting place.

1.4 Bifurcation

Fixed points can change stability or can be created/destroyed depending on changes in parameters in the problem. These changes are called *bifurcations*. The parameter values at which they occur are called *bifurcation points*. A *saddle–node bifurcation* is a bifurcation in which two fixed points are destroyed or created. A *transcritical bifurcation* is a bifurcation in which two fixed points exchange stability as they pass through one another.

Definition 1.11 (Bifurcation Diagram). The bifurcation diagram displays the location of fixed points as a function of the parameter.

We regard f as a function of both x and λ and examine

$$\dot{x} = f(x, \lambda)$$

near a bifurcation at $\lambda = \lambda_c$ and $x = x^*$ (a fixed point). We use Taylor's series around (x^*, λ_c)

$$\dot{x} = f(x,\lambda) = f(x^*,\lambda_c) + (x - x^*)\left.\frac{\partial f}{\partial x}\right|_{(x^*,\lambda_c)} + (\lambda - \lambda_c)\left.\frac{\partial f}{\partial \lambda}\right|_{(x^*,\lambda_c)} + \frac{1}{2}(x - x^*)^2\left.\frac{\partial^2 f}{\partial x^2}\right|_{(x^*,\lambda_c)} + \cdots$$

$f(x^*, \lambda_c) = 0$ (fixed point),
$\left.\frac{\partial f}{\partial x}\right|_{(x^*,\lambda_c)} = 0$ (tangency condition of local bifurcation).
Let

$$a = \left.\frac{\partial f}{\partial \lambda}\right|_{(x^*,\lambda_c)}, \quad b = \left.\frac{\partial^2 f}{\partial x^2}\right|_{(x^*,\lambda_c)}.$$

Then

$$\dot{x} = a(\lambda - \lambda_c) + \frac{b}{2}(x - x^*)^2,$$

where we neglect quadratic terms in $(\lambda - \lambda_c)$ and cubic terms in $(x - x^*)$. The normal form for the saddle–node bifurcation is

$$\dot{x} = \lambda + x^2.$$

The normal form for a transcritical bifurcation is

$$\dot{x} = \lambda x - x^2.$$

The normal form for a supercritical pitchfork bifurcation is

$$\dot{x} = \lambda x - x^3.$$

The normal form for a subcritical pitchfork bifurcation is

$$\dot{x} = \lambda x + x^3.$$

1.5 Dynamics on a Circle

Consider a differential equation whose phase space is a circle.

$$\dot{\theta} = f(\theta), \quad \theta \in [0, L) =: S_L,$$

where S_L is a circle of circumference $L > 0$. A vector field on the circle is a rule assigning a *unique* velocity to each point. A nonuniform oscillator is

$$\dot{\theta} = \omega - a\sin\theta, \quad \omega > 0, \quad a \geq 0.$$

2 Two-Dimensional Flows

2.1 Linear Systems and Eigenvalue Analysis

Consider

$$\dot{\mathbf{x}} = A\mathbf{x}, \quad \mathbf{x} \in \mathbb{R}^2.$$

$$\det(A - \lambda I) = \det\begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix} = \lambda^2 - (a + d)\lambda + (ad - bc).$$

Define

$$\tau = \operatorname{tr}(A) = a + d, \quad \Delta = \det(A) = ad - bc.$$

Then the characteristic polynomial is

$$\lambda^2 - \tau\lambda + \Delta = 0,$$

and

$$\lambda_{1,2} = \frac{1}{2}\left(\tau \pm \sqrt{\tau^2 - 4\Delta}\right).$$

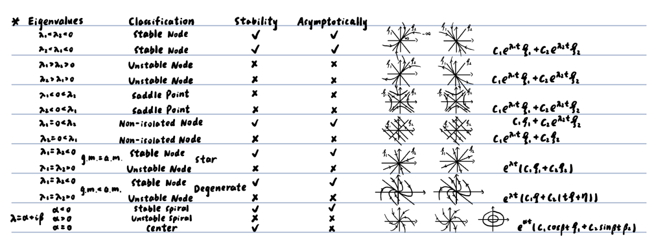
If A has two possibly complex eigenvalues λ_1, λ_2 with corresponding eigenvectors $\mathbf{v}_1, \mathbf{v}_2$ in the real distinct-eigenvalue case, then

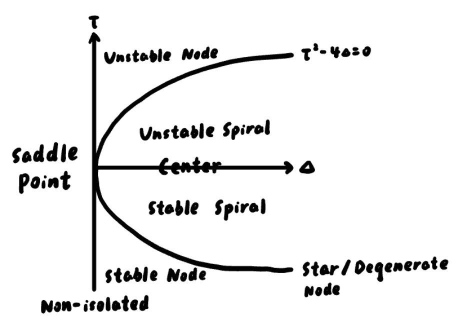
$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2, \quad c_1, c_2 \in \mathbb{R}.$$

$$\lambda_{1,2} = \alpha \pm i\beta, \quad \alpha = \operatorname{Re}(\lambda_{1,2}),\ \beta = \operatorname{Im}(\lambda_1) = -\operatorname{Im}(\lambda_2),\ \beta \neq 0.$$

$$\mathbf{x}(t) = c_1 e^{(\alpha + i\beta)t}\mathbf{v}_1 + c_2 e^{(\alpha - i\beta)t}\mathbf{v}_2 = e^{\alpha t}\left(c_1 e^{i\beta t}\mathbf{v}_1 + c_2 e^{-i\beta t}\mathbf{v}_2\right).$$

- If $\lambda_1, \lambda_2 \in \mathbb{R}$ with $\lambda_1 > \lambda_2 > 0$. Then $\mathbf{0}$ is an unstable node.
- If $\lambda_1, \lambda_2 \in \mathbb{R}$ with $\lambda_2 < \lambda_1 < 0$. Then $\mathbf{0}$ is a stable node. Trajectories approach tangent to the *slow eigendirection*.
- If $\lambda_1, \lambda_2 \in \mathbb{R}$ with $\lambda_2 < 0 < \lambda_1$. Then $\mathbf{0}$ is a saddle point. *Stable manifold* is span$\{\mathbf{v}_2\}$, and *unstable manifold* is span$\{\mathbf{v}_1\}$.
- If $\lambda_1 = \lambda_2 = \lambda \in \mathbb{R}$ and there are two linearly independent eigenvectors, then $\mathbf{0}$ is a *star node*.
- If $\lambda_1 = \lambda_2 = \lambda \in \mathbb{R}$ and there is only one linearly independent eigenvector, then $\mathbf{0}$ is a *degenerate node*.
- If $\alpha = 0$, then $\mathbf{0}$ is a *center* (neutrally stable).
- If $\alpha > 0$, then $\mathbf{0}$ is an unstable spiral.
- If $\alpha < 0$, then $\mathbf{0}$ is a stable spiral.





2.2 Nonlinear Two-Dimensional System

Theorem 2.1 (Existence and Uniqueness). *Consider the initial value problem*

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x}(0) = \mathbf{x}_0.$$

Assume $\mathbf{f}$ *is continuous and all partial derivatives*

$$\frac{\partial f_i}{\partial x_j}, \quad i, j = 1, \ldots, n$$

are continuous for $\mathbf{x}$ *in some open connected set* $D \subset \mathbb{R}^n$*. Then for* $\mathbf{x}_0 \in D$*, there exists* $\varepsilon > 0$ *such that a solution* $\mathbf{x}(t)$ *exists on* $(-\varepsilon, \varepsilon)$ *and is unique.*

For the planar system

$$\dot{x} = f(x, y), \quad \dot{y} = g(x, y),$$

a *fixed point* (x^*, y^*) satisfies

$$f(x^*, y^*) = 0, \quad g(x^*, y^*) = 0.$$

Let

$$u = x - x^*, \quad v = y - y^*$$

be a small disturbance. Then

$$\dot{u} = \dot{x} = f(x^* + u, y^* + v), \quad \dot{v} = \dot{y} = g(x^* + u, y^* + v).$$

$$f(x^* + u, y^* + v) = f(x^*, y^*) + u \left.\frac{\partial f}{\partial x}\right|_{(x^*, y^*)} + v \left.\frac{\partial f}{\partial y}\right|_{(x^*, y^*)} + O(u^2, v^2, uv),$$

$$g(x^* + u, y^* + v) = g(x^*, y^*) + u \left.\frac{\partial g}{\partial x}\right|_{(x^*, y^*)} + v \left.\frac{\partial g}{\partial y}\right|_{(x^*, y^*)} + O(u^2, v^2, uv).$$

Since (x^*, y^*) is a fixed point, $f(x^*, y^*) = g(x^*, y^*) = 0$, so

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = A \begin{pmatrix} u \\ v \end{pmatrix} + \text{(quadratic terms)},$$

where the *Jacobian matrix* at (x^*, y^*) is

$$A = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}_{(x^*, y^*)}.$$

Neglecting quadratic terms yields the *linearized system*

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = A \begin{pmatrix} u \\ v \end{pmatrix}.$$

2.3 Conservative systems

Definition 2.2. Fixed points s.t. $\operatorname{Re} \lambda \neq 0$ are called *hyperbolic.*

Definition 2.3 (Conserved Quantity). Given a system

$$\dot{x} = f(x),$$

a conserved quantity is a real-valued continuous function $I(x)$ that is constant on trajectories, i.e.

$$\frac{dI}{dt} = 0.$$

To avoid trivial examples, we also require that $I(x)$ be nonconstant on every open set.

Definition 2.4 (Conservative Systems). Systems for which a conserved quantity exists are called *conservative systems.*

Theorem 2.5. *A conservative system cannot have any attracting fixed points.*

Proof. Suppose x^* was an attracting fixed point. Then all points in its basin of attraction would have to be at the same energy $E(x^*)$, because energy is constant on trajectories. Hence, $E(x)$ must be a constant function in the basin of attraction of x^*. But this contradicts our definition of a conservative system, in which we require that $E(x)$ be nonconstant on all open sets. □

Theorem 2.6 (Nonlinear centers for conservative systems). *Consider the system*

$$\dot{x} = f(x), \quad x \in \mathbb{R}^2,$$

and suppose f *is continuously differentiable. Suppose there exists a conserved quantity* $I(x)$ *and suppose that* x^* *is an isolated fixed point. If* x^* *is a local minimum of* I*, then all trajectories sufficiently close to* x^* *are closed.*

2.4 Limit Cycle

Definition 2.7 (Limit Cycle). A *limit cycle* is an isolated closed trajectory. Isolated means that neighboring trajectories are not closed. They spiral either toward or away from the limit cycle. Limit cycles are inherently nonlinear phenomena. They cannot occur in linear systems. A linear system can have closed orbits, but they will not be isolated.

Definition 2.8. If all neighboring trajectories approach the limit cycle, we say the limit cycle is *stable* (or *attracting*). Otherwise, the limit cycle is *unstable*, or *half-stable*.

Definition 2.9 (Gradient System). Suppose the system can be written in the form

$$\dot{x} = -\nabla V(x),$$

for some continuously differentiable, single-valued function $V(x)$. Such a system is called a *gradient system* with potential function V.

Theorem 2.10. *Closed orbits are impossible in gradient systems.*

Definition 2.11. Let

$$\dot{x} = f(x, y), \quad \dot{y} = g(x, y)$$

be a smooth vector field on the phase plane. The system is a gradient system if and only if

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}.$$

Theorem 2.12 (Bendixson's Theorem). *If a region* $D \subset \mathbb{R}^2$ *has no holes and*

$$\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} \neq 0 \quad \text{for all } (x, y) \in D,$$

and does not change sign in D*, then the system*

$$\begin{cases} \dot{x} = f(x, y), \\ \dot{y} = g(x, y) \end{cases}$$

has no closed trajectories in D*.*

2.5 Lyapunov Function

Theorem 2.13 (Lyapunov Function). *Consider a system*

$$\dot{x} = f(x)$$

with a fixed point x^*. Suppose that we can find a Lyapunov function $V(x)$, i.e., a continuously differentiable real-valued function $V(x)$ with the following properties

(1) $V(x) > 0$ *for all* $x \neq x^*$ *and* $V(x^*) = 0$.
(2) $\dot{V}(x) \leq 0$ *for all* $x \neq x^*$.

Then x^* is globally asymptotically stable for all initial conditions, and

$$x(t) \to x^* \quad \text{as } t \to \infty.$$

In particular, the system has no closed orbits.

Theorem 2.14 (Dulac's Criterion). *Let $\dot{x} = f(x)$ be a continuously differentiable vector field defined on a simply connected subset R of the plane. If there exists a continuously differentiable real-valued function $g(x)$ such that*

$$\nabla \cdot (gf)$$

has one sign throughout R, then no closed orbits lie entirely in R.

Theorem 2.15 (Poincaré–Bendixson). *Suppose that*

(1) *R is a closed, bounded subset of the plane.*
(2) *$\dot{x} = f(x)$ is a continuously differentiable vector field on an open set containing R.*
(3) *R does not contain any fixed points.*
(4) *There exists a trajectory C that is confined in R, i.e., it starts in R and stays in R for all future time.*

Then either C is a closed orbit or it spirals toward a closed orbit as $t \to \infty$. In either case, R contains a closed orbit.

2.6 Weakly Nonlinear Oscillators

Definition 2.16. An equation of the form

$$x'' + x + \varepsilon f(x, x') = 0$$

is called a *weakly nonlinear oscillator.*

Definition 2.17 (Regular Perturbation Theory).

$$x(t) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \cdots.$$

$$x' = x_0'(t) + \varepsilon x_1'(t) + \varepsilon^2 x_2'(t) + \cdots, \quad x'' = x_0''(t) + \varepsilon x_1''(t) + \varepsilon^2 x_2''(t) + \cdots.$$

Plugging into the governing ODE and grouping terms by powers of ε gives a hierarchy of linear problems for $\{x_k\}$.

Definition 2.18 (Two Time Scales (Method of Multiple Scales)). Long-time changes occur slowly compared with fast oscillations. Introduce

$$\tau = t \text{ (fast time)}, \quad T = \varepsilon t \text{ (slow time)}.$$

$$\dot{x} = \frac{\mathrm{d}x}{\mathrm{d}t} = x_\tau + \varepsilon x_T, \quad \ddot{x} = \frac{\partial^2 x}{\partial \tau^2} + 2\varepsilon \frac{\partial^2 x}{\partial \tau \partial T} + \varepsilon^2 \frac{\partial^2 x}{\partial T^2}.$$

$$x(\tau, T; \varepsilon) = x_0(\tau, T) + \varepsilon x_1(\tau, T) + O(\varepsilon^2).$$

$$\dot{x} = x_{0\tau} + \varepsilon\big(x_{1\tau} + x_{0T}\big) + O(\varepsilon^2), \quad \ddot{x} = x_{0\tau\tau} + \varepsilon\big(x_{1\tau\tau} + 2x_{0\tau T}\big) + O(\varepsilon^2).$$

Definition 2.19 (Van der Pol Oscillator and Limit Cycle). Consider the Van der Pol oscillator

$$\ddot{x} + x + \varepsilon(x^2 - 1)\dot{x} = 0,$$

with $\tau = t$, $T = \varepsilon t$.

$$x_{0\tau\tau} + x_0 = 0 \implies x_0(\tau, T) = R(T)\cos\big(\tau + \phi(T)\big).$$

$$x_{1\tau\tau} + x_1 = -2x_{0\tau T} - (x_0^2 - 1)x_{0\tau} = \left(2R' - R + \frac{1}{4}R^3\right)\sin\theta + 2R\phi'\cos\theta + \frac{1}{4}R^3\sin(3\theta).$$

To avoid resonant forcing (the $\sin\theta$ and $\cos\theta$ terms),

$$2R\phi' = 0, \quad 2R' - R + \frac{1}{4}R^3 = 0.$$

For nontrivial oscillations ($R \neq 0$),

$$\phi'(T) = 0 \implies \phi(T) = \phi_0 \text{ (constant)},$$

and

$$R' = \frac{1}{8}R(4 - R^2), \quad R \geq 0.$$

$$R^* = 0 \quad \text{(unstable)}, \quad R^* = 2 \quad \text{(stable)}.$$

Therefore $R(T) \to 2$ as $T \to \infty$,

$$x_0(t, T) \to 2\cos\big(t + \phi_0\big).$$

Thus solutions approach a stable limit cycle with radius

$$r = 2 + O(\varepsilon).$$

Let $\theta = t + \phi(T)$. Then the angular frequency is

$$\omega = \frac{\mathrm{d}\theta}{\mathrm{d}t} = 1 + \varepsilon\phi'(T) = 1 + O(\varepsilon^2).$$

2.7 2D Bifurcation

$$\text{Saddle-node.} \begin{cases} \dot{x} = \mu - x^2, \\ \dot{y} = -y. \end{cases}$$

$$\text{Transcritical.} \begin{cases} \dot{x} = \mu x - x^2, \\ \dot{y} = -y. \end{cases}$$

$$\text{Supercritical pitchfork.} \begin{cases} \dot{x} = \mu x - x^3, \\ \dot{y} = -y. \end{cases}$$

$$\text{Subcritical pitchfork.} \begin{cases} \dot{x} = \mu x + x^3, \\ \dot{y} = -y. \end{cases}$$

The saddle-node, transcritical, and pitchfork bifurcations are all zero-eigenvalue bifurcations.

For a planar system, the linearization at an equilibrium can have a complex conjugate pair

$$\lambda_{1,2} \in \mathbb{C}.$$

Suppose a is a parameter and a_c denotes the critical value where bifurcation occurs.

(1) For $a < a_c$: stable spiral, attracted to the fixed point.
(2) For $a > a_c$: unstable spiral. Since bifurcations are local, the far-field dynamics should be essentially unchanged. Thus, far away trajectories move inward, but near the fixed point they move outward. So, a **limit cycle** has been created.

Supercritical Hopf bifurcation: a stable limit cycle exists for $a > a_c$.

$$\begin{cases} \dot{r} = \mu r - r^3, & \mu \text{ is the bifurcation parameter,} \\ \dot{\theta} = \omega + r^2, & \omega > 0. \end{cases}$$

In the subcritical case, after bifurcation, trajectories must jump to a distant attractor, which may be a fixed point, another limit cycle, or in dimension ≥ 3 a chaotic attractor.

$$\begin{cases} \dot{r} = \mu r + r^3 - r^5, & \mu \text{ is the bifurcation parameter,} \\ \dot{\theta} = \omega + r^2, & \omega > 0. \end{cases}$$

3 Chaos

3.1 Discrete Map

Definition 3.1 (Discrete Map). Consider a class of dynamical systems in which time is discrete. These systems are known as difference equations, recursion relations, iterated maps, or simply maps. A difference equation system has the general form

$$x_{n+1} = f(x_n, n),$$

where $n = 0, 1, 2, 3, \dots = \mathbb{N}_0$.

Definition 3.2 (Fixed Points). A fixed point of a difference system is a point whose further iteration does not change. That is,

$$x_{n+1} = x_n.$$

Thus fixed points satisfy

$$x^* = f(x^*).$$

Theorem 3.3 (Nonlinear 1D Map and Linear Stability Analysis). *Suppose x^* is a fixed point of*

$$x_{n+1} = f(x_n),$$

and f is nonlinear in x_n. Let

$$x_n = x^* + \varepsilon_n,$$

where ε_n is the deviation from x^*.

$$x^* + \varepsilon_{n+1} = f(x^* + \varepsilon_n).$$

Since x^* is a fixed point, $f(x^*) = x^*$, hence

$$\varepsilon_{n+1} = f'(x^*)\varepsilon_n + O(\varepsilon_n^2).$$

- *If $|f'(x^*)| < 1$, then x^* is stable.*
- *If $|f'(x^*)| > 1$, then x^* is unstable.*
- *If $|f'(x^*)| = 1$, linear stability analysis is inconclusive.*

Cobweb diagrams allow us to iterate the map graphically.

(1) Draw a vertical line from x_0 to the graph of $y = f(x)$.
(2) Move horizontally to the diagonal line $y = x$.
(3) Move vertically again to the curve.
(4) Repeat the process to generate successive iterates.

3.2 Logistic Map

Definition 3.4 (Periodic Orbit). A periodic orbit of period N is a sequence of N points

$$(x_0, x_1, \dots, x_{N-1})$$

such that

$$x_N = x_0.$$

Period-N orbits satisfy

$$x = f^N(x) = f(f(\cdots f(x)\cdots)).$$

Definition 3.5 (Logistic Map).

$$x_{n+1} = rx_n(1 - x_n)$$

Here $x_n > 0$ is a dimensionless measure of the population and $r > 0$ is the growth rate. At $r = 1$, transcritical bifurcation. At $r = 3$, period-doubling bifurcation. Solving $x = f^2(x)$ gives

$$x = 0, \quad x = 1 - \frac{1}{r},$$

and

$$x_\pm = \frac{1}{2}\left(1 + \frac{1}{r} \pm \sqrt{\frac{(r-1)(r-3)}{r}}\right).$$

$x = 0$ and $x = 1 - \frac{1}{r}$ exist for all r. These are fixed points of the logistic map (1-cycles). Fixed points are N-cycles for all N. The new solutions exist when

$$(r-1)(r-3) > 0.$$

When $r > 3$, the fixed point becomes unstable and the new 2-cycle (x_+, x_-) appears. This is called a *period-doubling bifurcation*. To investigate the stability of the period-2 orbit, we examine the stability of the orbit as a fixed point of

$$x_{n+1} = F(x_n), \quad F(x) = f(f(x)) = f^2(x).$$

The period-2 cycle is stable if $|F'(x^\star)| < 1$, which yields

$$3 < r < 1 + \sqrt{6}.$$

At

$$r > 1 + \sqrt{6},$$

the period-2 cycle becomes unstable and a period-4 cycle appears. A period-doubling bifurcation occurs when a stable period-N orbit loses stability and gives rise to a stable period-$2N$ orbit. There is a period 2^n orbit in this system for all $n \in \mathbb{N}$. Let r_n denote the value of r where the period-2^n cycle first appears. It can be shown that

$$r_1 = 3, \quad r_2 = 3.449, \quad r_3 = 3.54409,$$

$$r_\infty = \lim_{n\to\infty} r_n = 3.569946\dots$$

The successive bifurcations come faster and faster. The convergence is essentially geometric. In the limit of large n, the distance between successive period-doubling bifurcations shrinks by a constant factor

$$\delta = \lim_{n\to\infty} \frac{r_n - r_{n-1}}{r_{n+1} - r_n} = 4.669\dots$$

δ is called the *Feigenbaum constant*, which is a universal irrational constant for unimodal maps. This property is called *universality*. When $r > r_\infty$, the long-term behavior is aperiodic. The dynamics is all over the place, called chaotic.

3.3 Chaos

Definition 3.6 (Strogatz Definition). Chaos is aperiodic long–term behavior in a deterministic system that exhibits sensitive dependence on initial conditions (SDIC).

- **Aperiodic** means that there are trajectories which do not settle down to fixed points or periodic orbits.
- **Deterministic** means that the system has no random or noisy inputs or parameters.
- **SDIC** means nearby trajectories separate exponentially fast.

Definition 3.7 (Devaney Definition). The dynamics on a compact set S is chaotic provided

(1) It is transitive on the set S (orbits get arbitrarily close to every point in the set).
(2) It depends sensitively on initial conditions (SDIC).

Definition 3.8 (Periodic windows). After stretches of chaotic behavior, there appear to be windows of r values where the behavior is regular.

Definition 3.9. Suppose a map $f(x)$ is defined on $[0,1] \longrightarrow [0,1]$. Such a map $f(x)$ is called *unimodal* if it is smooth, concave down, and has a single maximum.

Theorem 3.10. *Let $f(x)$ be unimodal and suppose that*

$$x_{n+1} = f(x_n)$$

has an unstable period–3 orbit. Then the map has unstable periodic orbits of any integer period. The difference equation is chaotic.

3.4 Fractals

Definition 3.11 (Fractals). Fractals are geometric structures with fine structure at arbitrarily small scales. No matter how much zooming we do, we always see more features. Often fractals have some degree of self–similarity. The new features that appear upon zooming in are partially or wholly repetitions of the original structure.

Definition 3.12 (Set). A set is a collection of things called elements.

Definition 3.13 (Cardinality). Two sets X and Y are said to have the same cardinality or number of elements if there is an invertible mapping that pairs each element $x \in X$ with precisely one $y \in Y$. Such a mapping is called a one-to-one correspondence.

Definition 3.14 (Countable). A set with an infinite number of elements is countable if its elements can be put in one-to-one correspondence with the natural numbers. Otherwise, the set is uncountable.

Definition 3.15 (Cantor Sets).

$$C = \bigcap_{n=0}^{\infty} C_n.$$

- C is a collection of points.
- C is fractal.
- C is self-similar.
- C is uncountable (Cantor's diagonal argument).
- The Lebesgue measure is

$$\text{measure}(C) = 1 - \sum_{n=0}^{\infty} \frac{2^n}{3^{n+1}} = 1 - 1 = 0.$$

- The dimension (Hausdorff dimension) is

$$d = \frac{\ln 2}{\ln 3} \approx 0.63.$$

- C is totally disconnected.
- C has no isolated points (every point in C has another point in C arbitrarily close to it).

Definition 3.16 (Similarity Dimension). Suppose that a self-similar set consists of M copies of itself scaled down by a factor of R. We define the similarity dimension d of this set as

$$d = \frac{\ln M}{\ln R}.$$

Definition 3.17 (Box-Counting Dimension). Measure the set in a way that ignores irregularities of size less than ε, and then study how measurements vary as $\varepsilon \longrightarrow 0$. Suppose S is a subset of $\mathbb{R}^D$. Let $N(\varepsilon)$ be the minimal number of D-dimensional cubes of side ε that we need to completely cover S. The box-counting dimension of S is defined to be

$$d = \lim_{\varepsilon \to 0} \frac{\ln N(\varepsilon)}{\ln(1/\varepsilon)}.$$

Definition 3.18 (Lyapunov Exponents). Assume

$$|f^N(x_0 + \delta) - f^N(x_0)| \approx |\delta| e^{\lambda N}, \quad N \gg 1.$$

Thus,

$$\lambda = \frac{1}{N} \ln \left(\frac{|f^N(x_0 + \delta) - f^N(x_0)|}{|\delta|} \right).$$

As $\delta \longrightarrow 0$,

$$\lambda = \frac{1}{N} \ln \left| \frac{d}{dx} f^N(x_0) \right| = \frac{1}{N} \ln \left| \prod_{i=0}^{N-1} f'(x_i) \right| = \frac{1}{N} \sum_{i=0}^{N-1} \ln |f'(x_i)|.$$

Define the limit to be the Lyapunov exponent

$$\lambda = \lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} \ln |f'(x_i)|$$

for the orbit starting at x_0. λ is the same for all x_0 in the basin of attraction of a given attractor. For stable orbits, $\lambda < 0$ since $|f'(x)| < 1$. For unstable orbits, the Lyapunov exponent is positive. A positive Lyapunov exponent is a signature of chaos.

3.5 Lorenz Equations and Strange Attractors

Definition 3.19 (Attractor). Define an attractor to be a closed set A with the following properties

(1) A is invariant. Any trajectory $x(t)$ that starts in A stays in A.
(2) A attracts an open set of initial conditions. There exists an open set U containing A such that if $x(0) \in U$, then

$$\text{dist}(x(t), A) \longrightarrow 0 \quad \text{as } t \longrightarrow \infty.$$

The set U is called a *trapping region*. The largest such U is called the *basin of attraction* of A.

(3) A is minimal, the smallest set with the above two properties.

Definition 3.20 (Strange Attractor). A strange attractor is an attractor that exhibits sensitive dependence on initial data.

Definition 3.21 (Lorenz Equations).

$$\begin{cases} \dot{x} = \sigma(y - x), \\ \dot{y} = Rx - y - xz, \\ \dot{z} = xy - bz, \end{cases}$$

where $\sigma, R, b > 0$ are parameters. σ is the Prandtl number, R is the Rayleigh number, and b has no name. The butterfly-shaped set that trajectories go to in the Lorenz system is called the **Lorenz attractor**. The Lorenz attractor has box-counting dimension

$$d \approx 2.05.$$

Dynamics on the Lorenz attractor has Lyapunov exponent

$$\lambda \approx 0.9$$

hence neighboring trajectories separate exponentially fast. Thus, on the Lorenz attractor, the Lorenz system is chaotic.

Definition 3.22 (Volume contraction). Suppose a 3D system

$$\dot{\vec{x}} = \vec{F}(\vec{x}).$$

Let $S(t)$ be an arbitrary closed surface of volume $V(t)$ in phase space. Let $\vec{n}$ denote the outward normal on S. Since $\vec{F}$ is the instantaneous velocity of the points.

$$V(t + dt) = V(t) + \int_S (\vec{F} \cdot \vec{n})\, dt\, dA.$$

$$\dot{V} = \lim_{dt \to 0} \frac{V(t + dt) - V(t)}{dt} = \int_S (\vec{F} \cdot \vec{n})\, dA = \int_V (\nabla \cdot \vec{F})\, dV.$$

If $\dot{V}(t) < 0$, we say the system is volume contracting or dissipative. Dissipative differential equations have attractors. For Lorenz system

$$\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \vec{f}(\vec{x}) = \begin{pmatrix} \sigma(y - x) \\ rx - y - xz \\ xy - bz \end{pmatrix}.$$

$$\nabla \cdot f = \frac{\partial}{\partial x}[\sigma(y-x)] + \frac{\partial}{\partial y}[rx-y-xz] + \frac{\partial}{\partial z}[xy-bz] = -\sigma-1-b < 0.$$

The Lorenz system is dissipative. Volumes in phase space contract.

$$\dot{V} = (-\sigma - 1 - b)V \implies V(t) = V(0)e^{-(\sigma+1+b)t}.$$

Definition 3.23 (Lorenz Map).

$$z_{n+1} = f(z_n)$$

is called the **Lorenz map**, where z_n is nth local maximum of $z(t)$.

$$|f'(z)| > 1 \quad \text{for all } z.$$

$$\left| \frac{d}{dz} f^N(z_1) \right| = |f'(z_1)||f'(z_2)| \cdots |f'(z_N)| > 1.$$

Hence all periodic orbits are unstable.

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References

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