Introduction To Fluid Dynamics

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These notes are primarily based on the textbook by Kundu et al. [1].

1 Fluid

1.1 Solids, Liquids, and Gases

A fluid deforms continuously under any nonzero shear stress. A solid returns to a preferred shape when unloaded, if elastic. Liquids are nearly incompressible and form a free surface in gravity. Gases expand to fill their container.

1.2 Continuum Hypothesis

Although fluids are molecular, most macroscopic phenomena can be modeled by treating them as continua when the *Knudsen number* $\mathrm{Kn} = \ell/L \ll 1$, where ℓ is the molecular mean free path and L a characteristic length.
For a gas with mean velocity $\mathbf{u}$, the Maxwell velocity distribution is

$$f(\mathbf{v})\,\mathrm{d}^3v = n\left(\frac{m}{2\pi k_BT}\right)^{3/2}\exp\left(-\frac{m|\mathbf{v}-\mathbf{u}|^2}{2k_BT}\right)\mathrm{d}^3v,$$

with number density n, molecular mass m, temperature T, and Boltzmann constant k_B. When $\mathbf{u}=0$,

$$f(v) = \iint_{angles} f(\mathbf{v})v^2 d\Omega = 4\pi n\left(\frac{m}{2\pi k_BT}\right)^{3/2} v^2\exp\left(-\frac{mv^2}{2k_BT}\right),$$

with mean speed

$$\bar{v} = \frac{1}{n}\int_0^\infty v f(v)dv = \left(\frac{8k_BT}{\pi m}\right)^{1/2}.$$

The molecular mean free path ℓ (hard-sphere model, diameter d) is

$$\ell = \frac{1}{\sqrt{2}\,n\pi d^2}.$$

1.3 Molecular Transport Phenomena

Random molecular motion produces diffusive fluxes of species, heat, and momentum.

1.3.1 Species Diffusion (Fick's Law).
For mass fraction Y of a constituent in a mixture,

$$\mathbf{J}_m = -\rho\,k_m\nabla Y,$$

where $\mathbf{J}_m$ is mass flux, ρ is density, and k_m is the mass diffusivity.

1.3.2 Heat Conduction (Fourier's Law).

$$\mathbf{q} = -k\nabla T,$$

with heat flux $\mathbf{q}$ and thermal conductivity k.

1.3.3 Momentum Diffusion (Newton's Law of Viscosity).
For simple shear $u = u(y)$, the shear stress is

$$\tau = \mu\frac{\mathrm{d}u}{\mathrm{d}y},$$

with dynamic viscosity μ. The kinematic viscosity is

$$\nu \equiv \frac{\mu}{\rho}.$$

In gases, μ grows roughly as $T^{1/2}$ at fixed p. In liquids, μ generally decreases with T.

1.4 Surface Tension

At an interface, unbalanced molecular forces produce an effective *surface tension* σ (N/m), causing pressure jumps across curved surfaces (Laplace pressure). For a sphere of radius R,

$$p_{\text{in}} - p_{\text{out}} = \frac{2\sigma}{R}.$$

For a general interface with principal radii R_1, R_2,

$$p_{\text{in}} - p_{\text{out}} = \sigma\left(\frac{1}{R_1}+\frac{1}{R_2}\right).$$

1.5 Fluid Statics

Absolute pressure p relates to gauge pressure by $p_{\text{gauge}} = p - p_{\text{atm}}$. In a fluid at rest, shear stresses vanish and the normal stress is isotropic. Vertical force balance on a fluid element gives the hydrostatic relation

$$\frac{\mathrm{d}p}{\mathrm{d}z} = -\rho g.$$

The total force due to pressure acting on a surface A is

$$\vec{F} = \int_A -P\,\vec{n}\,dA,$$

where P is the scalar pressure field, $\vec{n}$ is the outward unit normal.

$$\vec{F} = \int_V \nabla\cdot(-P\mathbf{I})\,dV = \int_V -\nabla P\,dV.$$

Thus, the pressure force per unit volume is

$$\vec{f} = -\nabla P.$$

1.6 Classical Thermodynamics

1.6.1 First Law (per unit mass).

$$\mathrm{d}q + \mathrm{d}w = \mathrm{d}e.$$

For a reversible, quasi-static compression/expansion with specific volume $v = 1/\rho$ (volume per unit mass),

$$\mathrm{d}e = \mathrm{d}q - p\,\mathrm{d}v.$$

1.6.2 Equations of State.
For a simple compressible substance (single component), thermal and caloric equations of state is

$$p = p(v,T), \quad e = e(p,T).$$

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1.6.3 Enthalpy and Specific Heats.

$$h \equiv e + pv, \quad c_p \equiv \left(\frac{\partial h}{\partial T}\right)_p, \; c_v \equiv \left(\frac{\partial e}{\partial T}\right)_v .$$

The subscript p signifies that the derivative is taken at constant pressure. In reversible processes where the only work is $p\,\mathrm{d}v$,

$$\text{(const. pressure) } \mathrm{d}q = c_p\,\mathrm{d}T, \quad \text{(const. volume) } \mathrm{d}q = c_v\,\mathrm{d}T.$$

1.6.4 Second Law (Entropy). For a reversible path $1 \longrightarrow 2$,

$$s_2 - s_1 = \int_1^2 \frac{\mathrm{d}q_{\text{rev}}}{T}.$$

Clausius–Duhem inequality for arbitrary process,

$$s_2 - s_1 \geq \int_1^2 \frac{\mathrm{d}q}{T}.$$

Transport coefficients must be positive, species diffusivity $k_m > 0$, thermal conductivity $k > 0$, viscosity $\mu > 0$.

$$T\,\mathrm{d}s = \mathrm{d}q_{\text{rev}} = \mathrm{d}e + p\,\mathrm{d}v = \mathrm{d}h - v\,\mathrm{d}p.$$

For an ideal gas, $h = e + pv = c_v T + RT = c_p T$. Thus,

$$ds = c_p \frac{dT}{T} - R\frac{dp}{p}.$$

Integrating between (T, p) and (T_r, p_r),

$$\Delta s = c_p \ln\!\left(\frac{T}{T_r}\right) - R\ln\!\left(\frac{p}{p_r}\right).$$

Define the potential temperature θ as

$$\theta = T\left(\frac{p_0}{p}\right)^{\kappa}, \quad \kappa = \frac{R}{c_p}.$$

Substituting into the entropy difference expression yields

$$\Delta s = c_p \ln\!\left(\frac{\theta}{\theta_r}\right).$$

1.6.5 Speed of Sound and Thermal Expansion.

$$c^2 \equiv \left(\frac{\partial p}{\partial \rho}\right)_s \quad \alpha \equiv -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_p .$$

1.7 Perfect Gas

$$p = \rho RT, \quad R = c_p - c_v, \quad \gamma \equiv \frac{c_p}{c_v}.$$

Isentropic (adiabatic, frictionless) relations for constant c_p, c_v,

$$\frac{p}{\rho^\gamma} = \text{const}, \quad \frac{T}{T_0} = \left(\frac{p}{p_0}\right)^{(\gamma-1)/\gamma}, \quad \frac{\rho}{\rho_0} = \left(\frac{p}{p_0}\right)^{1/\gamma}.$$

Speed of sound and expansion coefficient,

$$c = \sqrt{\gamma RT}, \quad \alpha = \frac{1}{T}.$$

1.8 Stability of Stratified Fluid Media

We consider a fluid whose density ρ varies with height z. Stability is tested by displacing a small parcel vertically and examining the restoring buoyancy force.

1.8.1 Brunt–Väisälä (Buoyancy) Frequency. Define

$$N^2(z_0) = \frac{g}{\rho}\left(\frac{d\rho_a}{dz} - \frac{d\rho}{dz}\right)_{z_0}.$$

- $N^2 > 0$: parcel oscillates with frequency $N \Rightarrow$ stable.
- $N^2 = 0$: no restoring force $\Rightarrow$ neutral.
- $N^2 < 0$: exponential growth of displacement $\Rightarrow$ unstable.

Buoyancy (Acceleration)

$$b = -g\,\frac{\rho - \rho_0}{\rho} \approx g\,\frac{\rho_0 - \rho}{\rho_0},$$

where g is the gravitational acceleration, ρ is the parcel density, and ρ_0 is the ambient density. Under the ideal gas assumption,

$$b = -g\,\frac{T_0 - T}{T_0} = -g\,\frac{\theta_0 - \theta}{\theta_0}.$$

Thus, $b > 0$ implies upward acceleration of lighter/warmer fluid parcels, while $b < 0$ corresponds to downward acceleration of denser/colder parcels.

1.8.2 Potential Temperature θ. Define θ by adiabatically bringing a parcel at $(p(z), T(z))$ to a reference pressure p_0

$$T(z) = \theta(z)\left(\frac{p(z)}{p_0}\right)^{(\gamma-1)/\gamma}.$$

Differentiating with hydrostatic balance and perfect-gas law yields

$$\frac{T}{\theta}\frac{\mathrm{d}\theta}{\mathrm{d}z} = \frac{\mathrm{d}T}{\mathrm{d}z} + \frac{g}{c_p} \quad \equiv \; G - \Gamma_a,$$

where $G = \mathrm{d}T/\mathrm{d}z$ (lapse rate) and $\Gamma_a = -g/c_p$ (adiabatic lapse rate). Thus, stable if $\frac{\mathrm{d}\theta}{\mathrm{d}z} > 0$, neutral if $\frac{\mathrm{d}\theta}{\mathrm{d}z} = 0$, unstable if $\frac{\mathrm{d}\theta}{\mathrm{d}z} < 0$.

1.8.3 Potential Density ρ_θ. Define ρ_θ by adiabatically bringing the parcel to p_0

$$\rho(z) = \rho_\theta(z)\left(\frac{p(z)}{p_0}\right)^{1/\gamma}, \quad -\frac{1}{\rho_\theta}\frac{\mathrm{d}\rho_\theta}{\mathrm{d}z} = \frac{1}{\theta}\frac{\mathrm{d}\theta}{\mathrm{d}z} \;\Rightarrow\; \text{stable if } \frac{\mathrm{d}\rho_\theta}{\mathrm{d}z} < 0.$$

Oceanic Form (including compressibility). Using $c^{-2} = (\partial\rho/\partial p)_s$ and hydrostatic balance, a practical criterion is

$$\frac{\mathrm{d}\rho_\theta}{\mathrm{d}z} = \frac{\mathrm{d}\rho}{\mathrm{d}z} + \frac{\rho g}{c^2} \;\Rightarrow\; \text{stable if } \frac{\mathrm{d}\rho_\theta}{\mathrm{d}z} < 0.$$

1.8.4 Scale Height of the Atmosphere. With T constant and hydrostatic balance,

$$\frac{\mathrm{d}p}{\mathrm{d}z} = -\rho g = -\frac{pg}{RT} \;\Rightarrow\; p(z) = p_0 \exp\!\left(-\frac{z}{H}\right), \quad H \equiv \frac{RT}{g}.$$

2 Cartesian Tensors

2.1 Gradient, Divergence, and Curl

$$\nabla = \mathbf{e}_1\,\frac{\partial}{\partial x_1} + \mathbf{e}_2\,\frac{\partial}{\partial x_2} + \mathbf{e}_3\,\frac{\partial}{\partial x_3} = \sum_{i=1}^{3} \mathbf{e}_i\,\frac{\partial}{\partial x_i}.$$

The divergence $\nabla\cdot\mathbf{T}$ of a second-order tensor $\mathbf{T} = \{T_{ij}\}$ is the vector whose j-component is

$$(\nabla\cdot\mathbf{T})\cdot\mathbf{e}_j = \sum_{i=1}^{3}\frac{\partial T_{ij}}{\partial x_i}.$$

$$(\nabla\times\mathbf{u})\cdot\mathbf{e}_i = \sum_{j=1}^{3}\sum_{k=1}^{3}\varepsilon_{ijk}\,\frac{\partial u_k}{\partial x_j}\,.$$

A vector field $\mathbf{u}$ is called *solenoidal* (divergence free) if $\nabla\cdot\mathbf{u}=0$, and *irrotational* (curl free) if $\nabla\times\mathbf{u}=0$.

2.2 Gauss' Theorem

$$\iiint_V\left(\sum_{i=1}^{3}\frac{\partial Q_i}{\partial x_i}\right)dV = \iint_A\left(\sum_{i=1}^{3}n_iQ_i\right)dA = \iint_A\mathbf{n}\cdot\mathbf{Q}dA = \iiint_V(\nabla\cdot\mathbf{Q})dV.$$

$$\nabla\cdot Q = \lim_{V\to 0}\frac{1}{V}\iint_A\mathbf{n}\cdot Q\,dA,\quad \nabla\times Q = \lim_{V\to 0}\frac{1}{V}\iint_A\mathbf{n}\times Q\,dA.$$

2.3 Stokes' Theorem

$$\iint_A(\nabla\times\mathbf{u})\cdot\mathbf{n}\,\mathrm{d}A \;=\; \oint_C\mathbf{u}\cdot\mathbf{t}\,\mathrm{d}s.$$

The right-hand side is called the *circulation* of $\mathbf{u}$ about C.

$$\mathbf{n}\cdot(\nabla\times\mathbf{u}) \;=\; \lim_{A\to 0}\frac{1}{A}\oint_C\mathbf{u}\cdot\mathbf{t}\,\mathrm{d}s\,.$$

3 Kinematics

3.1 Particle and Field Descriptions of Fluid Motion

3.1.1 Lagrangian Description. Based on particle motion,

$$\boldsymbol{r}=\boldsymbol{r}(t;\boldsymbol{r}_0,t_0).$$

Here $\boldsymbol{r}_0$ and t_0 are boundary/initial–condition parameters.

$$\boldsymbol{u}=\frac{d\,\boldsymbol{r}(t;\boldsymbol{r}_0,t_0)}{dt},\quad \boldsymbol{a}=\frac{d^2\,\boldsymbol{r}(t;\boldsymbol{r}_0,t_0)}{dt^2}.$$

Any scalar, vector, or tensor field F may depend on the paths of the relevant fluid particles and on time, i.e. $F=F[\boldsymbol{r}(t;\boldsymbol{r}_0,t_0),\,t]$.

3.1.2 Eulerian Description. The Eulerian description focuses on properties at locations of interest and uses the four independent variables $(\boldsymbol{x},t)$ (three spatial coordinates and time). Thus a field quantity is written as $F=F(\boldsymbol{x},t)$.

3.1.3 Relating the Two Descriptions.

$$F[\boldsymbol{r}(t;\boldsymbol{r}_0,t_0),\,t]=F(\boldsymbol{x},t)\quad\text{when}\quad \boldsymbol{x}=\boldsymbol{r}(t;\boldsymbol{r}_0,t_0).$$

$$\frac{d}{dt}F[\boldsymbol{r}(t;\boldsymbol{r}_0,t_0),\,t]=\frac{\partial F}{\partial t}+\frac{\partial F}{\partial x_1}u_1+\frac{\partial F}{\partial x_2}u_2+\frac{\partial F}{\partial x_3}u_3=\frac{\partial F}{\partial t}+\boldsymbol{u}\cdot\nabla F\;\equiv\;\frac{DF(\boldsymbol{x},t)}{Dt},$$

which *defines* the Eulerian total material/substantial/particle derivative D/Dt. The first term, $\partial F/\partial t$, is the *unsteady* local rate of change at fixed $\boldsymbol{x}$, vanishing for steady fields. The second term, $\boldsymbol{u}\cdot\nabla F$, is the *advective* rate of change due to motion, vanishing when F is spatially uniform, the fluid is at rest, or $\boldsymbol{u}\perp\nabla F$.

3.2 Flow Lines, Fluid Acceleration, and Galilean Transformation

In the Eulerian description of fluid motion, three families of curves are commonly used: *streamlines*, *path lines*, and *streak lines*. Assume the velocity field $\boldsymbol{u}(\boldsymbol{x},t)$ is known for all $\boldsymbol{x}$ and t in the region of interest. When the flow is steady, these three curves coincide. In unsteady flows, they generally differ.

3.2.1 Streamlines. A streamline is a curve everywhere tangent to the instantaneous velocity field. If $\boldsymbol{ds}=(dx,dy,dz)$ is the arc-length element along a streamline and $\boldsymbol{u}=(u,v,w)$ is the local velocity, the tangency requirement gives

$$\frac{dx}{u}\;=\;\frac{dy}{v}\;=\;\frac{dz}{w},$$

and equivalently $\boldsymbol{u}\times\boldsymbol{ds}=\boldsymbol{0}$. If the seeds lie on a closed curve C, the swept surface forms a *stream tube*. No fluid crosses its mantle because $\boldsymbol{u}$ is everywhere tangent to it.

3.2.2 Path Lines. A path line is the trajectory of a material particle of fixed identity. Let $\boldsymbol{r}(t;\,\boldsymbol{r}_0,t_0)$ denote the position at time t of the particle that was at $\boldsymbol{r}_0$ at the reference time t_0. The path line satisfies

$$\frac{d\boldsymbol{r}}{dt}\;=\;\boldsymbol{u}(\boldsymbol{r}(t;\boldsymbol{r}_0,t_0),\,t),\quad \boldsymbol{r}(t_0;\boldsymbol{r}_0,t_0)=\boldsymbol{r}_0.$$

3.2.3 Streak Lines. A streak line at time t through a fixed point $\boldsymbol{x}_0$ is the locus of all particles that have passed or will pass through $\boldsymbol{x}_0$. Equivalently, $\boldsymbol{r}(t;\boldsymbol{x}_0,t_0)=\boldsymbol{x}$ with the condition $\boldsymbol{r}(t_0;\boldsymbol{x}_0,t_0)=\boldsymbol{x}_0$.

3.2.4 Material Acceleration and Galilean Invariance. The Eulerian field form of the material acceleration is

$$\frac{D\boldsymbol{u}}{Dt}\;=\;\frac{\partial\boldsymbol{u}}{\partial t}+(\boldsymbol{u}\cdot\boldsymbol{\nabla})\,\boldsymbol{u},$$

where the first term is the *unsteady* local acceleration and the second is the *advective* acceleration. The advective term is nonlinear (quadratic in $\boldsymbol{u}$) and vanishes if $\boldsymbol{u}=\boldsymbol{0}$ or if $\boldsymbol{u}$ is spatially uniform.

Consider two Cartesian frames with parallel axes. A stationary frame $Oxyz$ and a frame $O'x'y'z'$ translating with constant velocity $\boldsymbol{U}$ relative to $Oxyz$. If $\boldsymbol{u}$ and $\boldsymbol{u}'$ are the velocities observed in the two frames at corresponding locations and times, then

$$\boldsymbol{u}(\boldsymbol{x},t)\;=\;\boldsymbol{U}+\boldsymbol{u}'(\boldsymbol{x}',t'),\quad \boldsymbol{x}=\boldsymbol{x}'+\boldsymbol{U}t+\boldsymbol{x}_0',\quad t=t'.$$

Under this *Galilean transformation*, the material acceleration

$$\left.\frac{D\boldsymbol{u}}{Dt}\right|_{Oxyz}=\frac{\partial\boldsymbol{u}'}{\partial t}+(\boldsymbol{u}'\cdot\boldsymbol{\nabla}')\frac{\partial\boldsymbol{x}'}{\partial t}+(\boldsymbol{u}'\cdot\boldsymbol{\nabla}')\,(\boldsymbol{U}+\boldsymbol{u}')=\left.\frac{D\boldsymbol{u}'}{Dt'}\right|_{O'x'y'z'}.$$

3.3 Strain and Rotation Rates

Kinematically, the relative motion between neighboring points can be decomposed into parts due to local *deformation* and *rotation*. Let $\boldsymbol{u}(\boldsymbol{x},t)$ be the velocity at point O with position $\boldsymbol{x}$, and let $\boldsymbol{u}+\mathrm{d}\boldsymbol{u}$ be the velocity at a nearby point P at $\boldsymbol{x}+\mathrm{d}\boldsymbol{x}$.

$$\mathrm{d}u_i=\frac{\partial u_i}{\partial x_j}\,\mathrm{d}x_j.$$

The velocity gradient tensor decomposes uniquely into symmetric (strain-rate) and antisymmetric (rotation) parts.

$$\frac{\partial u_i}{\partial x_j}=S_{ij}+\tfrac{1}{2}R_{ij},\;\; S_{ij}=\tfrac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right),\;\; R_{ij}=\frac{\partial u_i}{\partial x_j}-\frac{\partial u_j}{\partial x_i}.$$

Here S_{ij} governs fluid-element deformation and is the quantity that couples to stress in the equations of motion, whereas R_{ij} represents rigid-body-like local rotation.

3.3.1 Volumetric Strain Rate. For a small control volume $\mathrm{d}V = \mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3$ carried with the fluid,

$$\frac{1}{\mathrm{d}V}\frac{\mathrm{D}}{\mathrm{D}t}(\mathrm{d}V) = \frac{1}{\mathrm{d}x_i}\frac{\mathrm{D}}{\mathrm{D}t}(\mathrm{d}x_i) = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = \frac{\partial u_i}{\partial x_i} = S_{ii},$$

which is $\nabla \cdot \boldsymbol{u}$ and is independent of coordinate orientation.

3.3.2 Shear Strain Rates. The average rate at which two material line segments initially parallel to x_i and x_j rotate toward or away from each other is

$$\frac{1}{2}\frac{\mathrm{D}(\alpha+\beta)}{\mathrm{D}t} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = S_{ij} \quad (i \neq j),$$

where α is angle by which the line initially parallel to x_i rotates toward x_j and β is angle by which the line initially parallel to x_j rotates toward x_i.

3.3.3 Rotation Tensor, Vorticity, and Irrotationality. The tensor R_{ij} corresponds to the *vorticity* vector $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$ via

$$R_{ij} = -\varepsilon_{ijk}\,\omega_k = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}.$$

Fluid motion is *irrotational* if

$$\boldsymbol{\omega} = \mathbf{0} \iff R_{ij} = \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} = 0,$$

in which case $\boldsymbol{u}$ can be represented as $\boldsymbol{u} = \nabla\phi$.

3.3.4 Circulation. The circulation Γ measures the net rotation content within a closed curve C.

$$\Gamma \equiv \oint_C \boldsymbol{u}\cdot \mathrm{d}\boldsymbol{s} = \iint_A \boldsymbol{\omega}\cdot\boldsymbol{n}\,\mathrm{d}A.$$

4 Conservation Laws

4.1 Conservation of Mass

Let $V(t)$ denote a *material volume*—the volume occupied by a specific collection of fluid particles. Such a volume moves and deforms with the flow so that it always contains the same mass elements. Consequently, the material surface $A(t)$ bounding $V(t)$ moves everywhere with the local fluid velocity $\boldsymbol{u}$.

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} \rho(\boldsymbol{x},t)\,\mathrm{d}V = 0.$$

Using the Reynolds transport theorem,

$$\int_{V(t)} \frac{\partial\rho(\boldsymbol{x},t)}{\partial t}\,\mathrm{d}V + \int_{A(t)} \rho(\boldsymbol{x},t)\,\boldsymbol{u}(\boldsymbol{x},t)\cdot\boldsymbol{n}\,\mathrm{d}A = 0.$$

This expresses the balance between the integrated density change within $V(t)$ and the integrated flux through its surface $A(t)$. Applying Gauss' divergence theorem,

$$\int_{V(t)} \left[\frac{\partial\rho(\boldsymbol{x},t)}{\partial t} + \nabla\cdot\big(\rho(\boldsymbol{x},t)\,\boldsymbol{u}(\boldsymbol{x},t)\big)\right]\mathrm{d}V = 0,$$

yielding the **continuity equation**

$$\frac{\partial\rho(\boldsymbol{x},t)}{\partial t} + \nabla\cdot\big(\rho(\boldsymbol{x},t)\,\boldsymbol{u}(\boldsymbol{x},t)\big) = 0.$$

$$\frac{1}{\rho(\boldsymbol{x},t)}\frac{D\rho(\boldsymbol{x},t)}{Dt} + \nabla\cdot\boldsymbol{u}(\boldsymbol{x},t) = 0.$$

For constant-density flow, and more generally, for *incompressible* flow,

$$\frac{D\rho}{Dt} \equiv \frac{\partial\rho}{\partial t} + \boldsymbol{u}\cdot\nabla\rho = 0, \quad \nabla\cdot\boldsymbol{u} = 0.$$

4.2 Conservation of Momentum

$$\frac{d}{dt}\int_{V(t)} \rho(\mathbf{x},t)\,\mathbf{u}(\mathbf{x},t)\,dV = \int_{V(t)} \rho(\mathbf{x},t)\,\mathbf{g}\,dV + \int_{A(t)} \mathbf{f}(\mathbf{n},\mathbf{x},t)\,dA.$$

Using the Reynolds transport theorem,

$$\int_{V(t)} \frac{\partial}{\partial t}(\rho\mathbf{u})\,dV + \int_{A(t)} \rho\mathbf{u}\,(\mathbf{u}\cdot\mathbf{n})\,dA = \int_{V(t)} \rho\mathbf{g}\,dV + \int_{A(t)} \mathbf{f}(\mathbf{n},\mathbf{x},t)\,dA.$$

For an arbitrarily moving control volume $V^*(t)$ with boundary $A^*(t)$ and control-surface velocity $\boldsymbol{\beta}(\mathbf{x},t)$, start from the RTT form

$$\int_{V^*(t)} \frac{\partial}{\partial t}(\rho\mathbf{u})\,dV = \frac{d}{dt}\int_{V^*(t)} \rho\mathbf{u}\,dV - \int_{A^*(t)} \rho\mathbf{u}\,(\boldsymbol{\beta}\cdot\mathbf{n})\,dA,$$

and choose $V^*(t)$ instantaneously coincident with $V(t)$ so that,

$$\frac{d}{dt}\int_{V^*(t)} \rho\mathbf{u}\,dV + \int_{A^*(t)} \rho\mathbf{u}\left[(\mathbf{u}-\boldsymbol{\beta})\cdot\mathbf{n}\right]dA = \int_{V^*(t)} \rho\mathbf{g}\,dV + \int_{A^*(t)} \mathbf{f}\,dA.$$

Choosing $\boldsymbol{\beta} = \mathbf{u}$ recovers the material-volume form.

4.2.1 Body and Surface Forces. Body forces act without contact. A conservative body force admits a potential Φ s.t.

$$\mathbf{g} = -\nabla\Phi$$

Surface forces act through contact and are expressed via the Cauchy stress tensor $\mathbf{T} = [T_{ij}]$. The traction (force per unit area) is

$$f_j = n_i T_{ij}.$$

4.2.2 Differential Form (Cauchy Momentum Equation).

$$\int_{A(t)} \rho\,\mathbf{u}\,(\mathbf{u}\cdot\mathbf{n})\,dA = \int_{V(t)} \nabla\cdot(\rho\,\mathbf{u}\mathbf{u})\,dV = \int_{V(t)} \frac{\partial}{\partial x_i}(\rho u_i u_j)\,dV,$$

$$\int_{A(t)} \mathbf{f}\,dA = \int_{A(t)} n_i T_{ij}\,dA = \int_{V(t)} \frac{\partial T_{ij}}{\partial x_i}\,dV.$$

$$\int_{V(t)} \left[\frac{\partial}{\partial t}(\rho u_j) + \frac{\partial}{\partial x_i}(\rho u_i u_j) - \rho g_j - \frac{\partial T_{ij}}{\partial x_i}\right]dV = 0.$$

$$\frac{\partial}{\partial t}(\rho u_j) + \frac{\partial}{\partial x_i}(\rho u_i u_j) = \rho g_j + \frac{\partial T_{ij}}{\partial x_i}.$$

Using continuity,

$$\frac{\partial\rho}{\partial t} + \frac{\partial(\rho u_i)}{\partial x_i} = 0,$$

giving the **Cauchy equation** of motion

$$\rho\frac{Du_j}{Dt} = \rho\,g_j + \frac{\partial T_{ij}}{\partial x_i}, \quad \frac{D}{Dt} = \frac{\partial}{\partial t} + u_i\frac{\partial}{\partial x_i}.$$

4.3 Constitutive Equation for a Newtonian Fluid

The stress at a point is specified by the nine components of the stress tensor T_{ij}, where the first index denotes the outward normal direction of the surface and the second index the direction in which the stress acts. The diagonal components T_{11}, T_{22}, T_{33} are normal stresses and the off–diagonal components are shear stresses. Considering the rotational dynamics of an infinitesimal fluid element shows that the stress tensor is symmetric,

$$T_{ij} = T_{ji},$$

so there are only six independent components. A *constitutive equation* relates stress and deformation. In a fluid at rest, stress is isotropic.

$$T_{ij} = -p\,\delta_{ij}.$$

When the fluid moves, additional viscous stresses σ_{ij} appear, both normal and shear.

$$T_{ij} = -p\,\delta_{ij} + \sigma_{ij}.$$

Galilean invariance requires σ_{ij} to depend on velocity *gradients*. Moreover, only shape change generates stress, so only the *symmetric* part of the velocity gradient.

$$\sigma_{ij} = K_{ijmn}\,S_{mn},$$

where K_{ijmn} is a fourth-order tensor that may depend on the local thermodynamic state. In an isotropic medium K_{ijmn} must be an isotropic tensor, which has the form

$$K_{ijmn} = \lambda\,\delta_{ij}\delta_{mn} + \mu\,\delta_{im}\delta_{jn} + \gamma\,\delta_{in}\delta_{jm}.$$

Symmetry of σ_{ij} implies K_{ijmn} is symmetric in (i, j), which forces

$$\gamma = \mu.$$

Thus,

$$\sigma_{ij} = 2\mu\,S_{ij} + \lambda\,S_{mm}\,\delta_{ij}, \quad S_{mm} \equiv \nabla\cdot\mathbf{u}.$$
$$T_{ij} = -p\,\delta_{ij} + 2\mu\,S_{ij} + \lambda\,S_{mm}\,\delta_{ij}.$$

Taking the trace,

$$T_{ii} = -3p + (2\mu + 3\lambda)\,S_{mm}.$$

Define the *mean* / mechanical pressure as

$$\bar{p} \;\equiv\; -\tfrac{1}{3}\,T_{ii},$$

so that

$$\bar{p} - p = \left(\tfrac{2}{3}\mu + \lambda\right)\nabla\cdot\mathbf{u}.$$

For compressible flow, the difference $\bar{p} - p$ relates to dilatation via the *bulk viscosity*

$$\mu_b \;\equiv\; \lambda + \tfrac{2}{3}\mu.$$

The Stokes assumption, $\lambda + \frac{2}{3}\mu = 0$, is often adequate when μ_b or the dilatation rate is small. Without invoking Stokes' assumption,

$$T_{ij} = -p\,\delta_{ij} + \sigma_{ij} = -p\,\delta_{ij} + 2\mu\big(S_{ij} - \tfrac{1}{3}S_{mm}\delta_{ij}\big) + \mu_b\,S_{mm}\,\delta_{ij}.$$

This linear relation reproduces $\sigma = \mu\,(\mathrm{d}u/\mathrm{d}y)$ for simple shear. Fluids are called *Newtonian.*

Non-Newtonian Behavior.

Power-Law Fluids. For a unidirectional shear $u = (u_1(x_2), 0, 0)$,

$$\sigma_{12} = \eta\,\dot{\gamma} = m\,\dot{\gamma}^{\,n}, \quad \dot{\gamma} \equiv \frac{\partial u_1}{\partial x_2},$$

with $n = 1$ Newtonian behavior, $n < 1$ (shear-thinning) and $n > 1$ (shear-thickening) common in polymeric and particulate systems.

Memory/Viscoelasticity. Linear viscoelastic responses can be written with a tensorial relaxation modulus $K_{ijmn}(t - t')$ as

$$\sigma_{ij}(t) = \int_{-\infty}^{t} K_{ijmn}(t - t')\,S_{mn}(t')\,\mathrm{d}t'.$$

Normal-Stress Differences in Shear. Even in simple shear, one may observe nonzero $T_{11} - T_{22}$ (first normal-stress difference) and $T_{22} - T_{33}$ (second normal-stress difference).

4.4 Navier-Stokes Momentum Equation

The momentum conservation equation for a Newtonian fluid

$$\rho\left(\frac{\partial u_j}{\partial t} + u_i\,\frac{\partial u_j}{\partial x_i}\right) = \rho\,g_j - \frac{\partial p}{\partial x_j} + \frac{\partial}{\partial x_i}\left[\mu\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right) + \left(\mu_v - \frac{2}{3}\mu\right)\left(\frac{\partial u_m}{\partial x_m}\right)\delta_{ij}\right].$$

When temperature differences within the flow are small,

$$\rho\,\frac{Du_j}{Dt} = \rho\,g_j - \frac{\partial p}{\partial x_j} + \mu\,\frac{\partial^2 u_j}{\partial x_i^2} + \left(\mu_v + \frac{1}{3}\mu\right)\frac{\partial}{\partial x_j}\left(\frac{\partial u_m}{\partial x_m}\right),$$

and, for incompressible flow, the vector form

$$\rho\,\frac{D\boldsymbol{u}}{Dt} = \rho\,\boldsymbol{g} - \nabla p + \mu\,\nabla^2\boldsymbol{u}.$$

For incompressible flow, the net viscous force per unit volume

$$\mu\,\frac{\partial^2 u_j}{\partial x_i^2} = \mu\,\frac{\partial}{\partial x_i}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right) = 2\mu\,\frac{\partial S_{ij}}{\partial x_i}$$
$$= -\,\mu\,\varepsilon_{jik}\,\frac{\partial \omega_k}{\partial x_i}, \text{ i.e., } \mu\,\nabla^2\boldsymbol{u} = -\,\mu\,\nabla\times\boldsymbol{\omega},$$

where S_{ij} is the strain-rate tensor and $\boldsymbol{\omega} = \nabla\times\boldsymbol{u}$ is the vorticity. Although rigid-body rotation does not appear in the Newtonian stress, *spatial derivatives* of the vorticity determine the viscous force. When viscous effects are negligible, e.g., many exterior flows away from solid boundaries, **Euler equation**

$$\rho\,\frac{D\boldsymbol{u}}{Dt} = \rho\,\boldsymbol{g} - \nabla p.$$

4.5 Special Forms of the Equations

4.5.1 Angular Momentum Principle for a Stationary Control Volume.

In solid mechanics one has

$$\frac{d\mathbf{H}}{dt} \;=\; \mathbf{M}, \quad \mathbf{H} \equiv \int_{V(t)} (\mathbf{r}\times\rho\mathbf{u})\,dV,$$

where $\mathbf{M}$ is the torque of external forces about a chosen axis, $\mathbf{r}$ is the position vector from that axis, and $\mathbf{u}$ is the velocity field. Applying the Reynolds transport theorem and specializing to a stationary control volume with fixed surface A_o and volume V_o gives

$$\frac{d}{dt}\int_{V_o}(\mathbf{r}\times\rho\mathbf{u})\,dV + \int_{A_o}(\mathbf{r}\times\rho\mathbf{u})\,(\mathbf{u}\cdot\mathbf{n})\,dA = \int_{V_o}(\mathbf{r}\times\rho\mathbf{g})\,dV + \int_{A_o}(\mathbf{r}\times\mathbf{f})\,dA.$$

4.5.2 Bernoulli Equations.

Consider inviscid flow with gravity as the only body force. The Euler equations are

$$\frac{\partial u_j}{\partial t} + u_i\,\frac{\partial u_j}{\partial x_i} \;=\; -\frac{1}{\rho}\,\frac{\partial p}{\partial x_j} \;-\; \frac{\partial \Phi}{\partial x_j}, \quad \Phi \equiv gz.$$

If the flow is *barotropic*, $\rho = \rho(p)$, then

$$\frac{\partial}{\partial x_j}\left(\int_{p_o}^{p}\frac{dp'}{\rho(p')}\right) = \frac{1}{\rho}\,\frac{\partial p}{\partial x_j}.$$

Using the vector identity

$$(\mathbf{u}\cdot\nabla)\,\mathbf{u} \;=\; -\,\mathbf{u}\times\boldsymbol{\omega} \;+\; \nabla\!\left(\frac{|\mathbf{u}|^2}{2}\right), \qquad \boldsymbol{\omega}\equiv\nabla\times\mathbf{u},$$

$$\frac{\partial\mathbf{u}}{\partial t} \;+\; \nabla\!\left[\frac{|\mathbf{u}|^2}{2} + \int_{p_o}^{p}\frac{dp'}{\rho(p')} + gz\right] \;=\; \mathbf{u}\times\boldsymbol{\omega}.$$

Steady Barotropic Inviscid Flow. If $\partial/\partial t = 0$, then

$$\nabla B \;=\; \mathbf{u}\times\boldsymbol{\omega}, \quad B \;\equiv\; \frac{|\mathbf{u}|^2}{2} + \int_{p_o}^{p}\frac{dp'}{\rho(p')} + gz,$$

so constant-B surfaces contain both streamlines and vortex lines, and along streamlines and vortex lines

$$\frac{|\mathbf{u}|^2}{2} + \int_{p_o}^{p}\frac{dp'}{\rho(p')} + gz \;=\; \text{constant}.$$

If the flow is also irrotational ($\boldsymbol{\omega} = 0$), then B is spatially uniform.

Unsteady Irrotational Barotropic Case. Let $\mathbf{u} = \nabla\phi$. Then,

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}\,|\nabla\phi|^2 + \int_{p_o}^{p}\frac{dp'}{\rho(p')} + gz \;=\; \text{constant}.$$

Bernoulli From the Energy Equation. For steady, inviscid, adiabatic flow with conservative body force and continuity used,

$$\rho u_i\frac{\partial}{\partial x_i}\!\left(e + \frac{|\mathbf{u}|^2}{2}\right) \;=\; \rho u_i g_i - \frac{\partial}{\partial x_j}(\rho u_j p)/\rho,$$

which reduces to the streamline form

$$h + \frac{|\mathbf{u}|^2}{2} + gz \;=\; \text{constant along streamlines}, \qquad h\equiv e + \frac{p}{\rho}.$$

Viscous, Constant-ρ, Irrotational Case. Starting from the incompressible NS equation and $\nabla\cdot\mathbf{u} = 0$ and $\boldsymbol{\omega} = 0$,

$$\rho\,\frac{D\mathbf{u}}{Dt} \;=\; \rho\mathbf{g} - \nabla p - \mu\,\nabla(\nabla\cdot\mathbf{u}) \;\;\Rightarrow\;\; \rho\frac{\partial\mathbf{u}}{\partial t} + \nabla\!\left(\frac{\rho|\mathbf{u}|^2}{2} + \rho gz + p\right) = \mathbf{0},$$

which, integrated along a streamline between points 1 and 2, gives

$$\int_1^2\frac{\partial\mathbf{u}}{\partial t}\cdot d\mathbf{s} \;+\; \left(\frac{|\mathbf{u}|^2}{2} + gz + \frac{p}{\rho}\right)_2 = \left(\frac{|\mathbf{u}|^2}{2} + gz + \frac{p}{\rho}\right)_1.$$

Equivalently, with $\mathbf{u} = \nabla\phi$,

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}\,|\nabla\phi|^2 + gz + \frac{p}{\rho} \;=\; \text{constant}.$$

4.5.3 Neglect of Gravity in Constant-Density Flows. Let p_s and ρ_s denote hydrostatic pressure and density satisfying $\mathbf{0} = \rho_s\mathbf{g} - \nabla p_s$.

$$\rho\,\frac{D\mathbf{u}}{Dt} \;=\; \rho'\,\mathbf{g} - \nabla p' \;+\; \mu\nabla^2\mathbf{u}, \quad p'\equiv p - p_s, \quad \rho'\equiv\rho-\rho_s.$$

For constant-density flow ($\rho' = 0$),

$$\rho\,\frac{D\mathbf{u}}{Dt} \;=\; -\,\nabla p' \;+\; \mu\nabla^2\mathbf{u}.$$

4.5.4 The Boussinesq Approximation. For low-Mach flows with small temperature-induced density variations and constant transport properties, one approximates the continuity equation by incompressibility $\nabla\cdot\mathbf{u} = 0$ and retains density variations only where multiplied by gravity.

$$\frac{D\mathbf{u}}{Dt} \;=\; \frac{\rho'}{\rho_0}\,\mathbf{g} - \frac{1}{\rho_0}\nabla p' \;+\; \nu\nabla^2\mathbf{u}, \quad \nu\equiv\frac{\mu}{\rho_0}, \quad \rho'\approx -\rho_0\alpha\,(T-T_0),$$

with thermal expansion coefficient α and reference state (ρ_0, T_0). For the energy balance,

$$\rho\,\frac{De}{Dt} \;=\; -\,p\,(\nabla\cdot\mathbf{u}) \;+\; \rho\,\varepsilon \;-\; \nabla\cdot\mathbf{q},$$

and, using $p = \rho RT$, $c_p - c_v = R$, and $\alpha = 1/T$ for a perfect gas, the compressional heating term $-\,p\,\nabla\cdot\mathbf{u}$ converts the left-hand side to a c_p-form. Neglecting viscous heating $\rho\varepsilon$ under Boussinesq scalings and using Fourier's law with constant k gives the temperature equation

$$\rho c_p\,\frac{DT}{Dt} \;=\; -\,\nabla\cdot\mathbf{q} \;\;\Rightarrow\;\; \frac{DT}{Dt} \;=\; \kappa\nabla^2 T, \quad \kappa\equiv\frac{k}{\rho c_p}.$$

Boussinesq Set. With $\mathbf{g} = -g\,\mathbf{e}_z$ and $\rho = \rho_0[1-\alpha(T-T_0)]$,

$$\nabla\cdot\mathbf{u} = 0, \quad \frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho_0}\nabla p' + \alpha g(T-T_0)\,\mathbf{e}_z + \nu\nabla^2\mathbf{u}, \quad \frac{DT}{Dt} = \kappa\nabla^2 T.$$

5 Compressible Flow

Compressible flows exhibit several nonintuitive phenomena compared with incompressible flows. Shock waves (near-discontinuities) may appear. An increase (or decrease) in area may accelerate (or decelerate) a uniform stream. Friction may increase a flow's speed. Heat addition may lower a flow's temperature. The importance of compressibility is characterized by the Mach number

$$M \;\equiv\; \frac{U}{c},$$

where U is a representative speed and c is the speed of sound, defined thermodynamically by $c^2 \;\equiv\; \left(\frac{\partial p}{\partial\rho}\right)_s$. Under isentropic conditions, the nondimensionalized continuity equation can be written as

$$\nabla\cdot\mathbf{u} \;=\; -\,M^2\left(\frac{\rho_0}{\rho}\right)\frac{D}{Dt}\left(\frac{p-p_0}{\rho_0 U^2}\right).$$

In engineering practice, flows with $M < 0.3$ are typically treated as incompressible. It shows $O(10\%)$ deviations from perfectly incompressible behavior when the remaining factors are order unity.

1. **Incompressible.** $M = 0$. Density does not vary with pressure in the flow field. A gas may be treated as constant-density.
2. **Subsonic.** $0 < M < 1$. No shock waves appear.
3. **Transonic.** $0.8 \lesssim M \lesssim 1.2$. Shock waves may appear. Analysis is difficult due to inherent nonlinearity and strong inviscid/viscous coupling.
4. **Supersonic.** $M > 1$. Shock waves are generally present. In some respects, analysis is easier since information propagates along characteristics whose directions can be determined.
5. **Hypersonic.** $M \gtrsim 3$. Very high speeds with friction or shocks can raise temperatures enough for molecular dissociation and other chemical effects.

5.1 Acoustics

Acoustics treats small, isentropic fluctuations of velocity, pressure, and density about steady reference values, providing the small–disturbance limit of compressible flow.

5.1.1 Governing Equations with Acoustic Sources.

$$\frac{1}{\rho}\frac{D\rho}{Dt} + \frac{\partial u_i}{\partial x_i} = q,$$
$$\frac{Du_j}{Dt} + \frac{1}{\rho}\frac{\partial p}{\partial x_j} = g_j + \frac{1}{\rho}\frac{\partial \sigma_{ij}}{\partial x_i} + f_j,$$

where $q(\mathbf{x},t)$ is a per-volume volume source, $f_j(\mathbf{x},t)$ is a per-mass body-force source, and g_j is the steady body force. In typical acoustic propagation, viscous stresses are negligible unless the frequency is very high or the propagation distance is very long.

5.1.2 Isentropic Thermodynamic Relation. For isentropic fluctuations following a fluid particle,

$$\frac{Dp}{Dt} = \left(\frac{\partial p}{\partial \rho}\right)_s \frac{D\rho}{Dt} \quad\Rightarrow\quad \frac{Dp}{Dt} = c^2\frac{D\rho}{Dt}.$$

5.1.3 Convected Wave Equation and Source Types. Neglecting viscosity for propagation and taking g_j uniform yields the convected wave operator acting on p with explicit acoustic sources.

$$\frac{D}{Dt}\left(\frac{1}{\rho c^2}\frac{Dp}{Dt}\right) - \frac{\partial}{\partial x_j}\left(\frac{1}{\rho}\frac{\partial p}{\partial x_j}\right) = \frac{Dq}{Dt} - \frac{\partial f_j}{\partial x_j} + \frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}.$$

The right-hand side represents, respectively, monopole (volume injection/expansion), dipole (divergence of fluctuating body force), and quadrupole (self-interaction of the flow) source terms. One may group them as a single scalar source $\dot{q}$.

$$\dot{q} \equiv \frac{Dq}{Dt} - \frac{\partial f_j}{\partial x_j} + \frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}.$$

5.1.4 Linearization About a Uniform State. Decompose fields into steady means and small fluctuations,

$$u_i = U_i + u_i', \quad p = p_0 + p', \quad \rho = \rho_0 + \rho', \quad T = T_0 + T',$$

and, for small isentropic variations,

$$p' = c^2\rho', \quad \frac{p'}{p_0} = \frac{p'}{\rho_0 c^2} \ll 1.$$

With U_i, p_0, ρ_0, T_0 uniform and time-invariant, the linearized, source-free pressure field satisfies

$$\frac{1}{c^2}\left(\frac{\partial}{\partial t} + U_i\frac{\partial}{\partial x_i}\right)^2 p' - \frac{\partial^2 p'}{\partial x_i\,\partial x_i} = 0.$$

5.1.5 Classical Wave Equation and Velocity–Pressure Relation. For a stationary medium ($U_i = 0$), the classical wave equation

$$\frac{1}{c^2}\frac{\partial^2 p'}{\partial t^2} - \frac{\partial^2 p'}{\partial x_i\,\partial x_i} = 0.$$

The linearized momentum equation relates acoustic velocity and pressure.

$$\frac{\partial u_j'}{\partial t} + \frac{1}{\rho_0}\frac{\partial p'}{\partial x_j} = 0 \quad\Rightarrow\quad u_j'(\mathbf{x},t) = -\frac{1}{\rho_0}\int \frac{\partial p'}{\partial x_j}\,dt.$$

5.1.6 One-Dimensional Solutions. For one-dimensional disturbances $p'(x,t)$ in a quiescent fluid, the solution is the d'Alembert form

$$p'(x,t) = f(x-ct) + g(x+ct),$$

and the accompanying particle velocity along x is

$$u'(x,t) = \frac{1}{\rho_0 c}\left[\, f(x-ct) - g(x+ct)\,\right].$$

With a uniform mean flow U in the $+x$ direction, the linear pressure solution becomes

$$p'(x,t) = f\big(x-(c+U)t\big) + g\big(x+(c-U)t\big),$$

so downstream-propagating waves convect at $c+U$ and upstream-propagating waves at $c-U$. When $U > c$ (supersonic), both travel downstream, altering causal influence.

5.1.7 On Weakly and Finitely Nonlinear Waves. The sound speed in an ideal gas depends on temperature, $c = \sqrt{\gamma RT}$. Because $\gamma > 1$, compressions ($p' > 0$) locally increase T and c, tending to steepen as they propagate, while expansions ($p' < 0$) decrease T and c, tending to spread. At sufficiently large amplitudes, compression waves form shocks and are no longer isentropic. They travel faster than linear acoustic waves in a still fluid.

6 Vorticity Dynamics

The *vorticity* $\boldsymbol{\omega} = \nabla\times\mathbf{u}$ is a vector field equal to twice the local angular velocity of a fluid particle. A localized concentration of nearly codirectional vorticity is called a *vortex*. Flows with circular or nearly circular streamlines are termed *vortex motions*.

A *vortex line* is everywhere tangent to the local vorticity vector, analogous to a streamline for the velocity field. If $\boldsymbol{\omega} = (\omega_x, \omega_y, \omega_z)$, an element $d\mathbf{s} = (dx, dy, dz)$ of a vortex line satisfies

$$\frac{dx}{\omega_x} = \frac{dy}{\omega_y} = \frac{dz}{\omega_z}.$$

Vortex lines do not exist in irrotational regions, just as streamlines do not exist in a fluid at rest. All vortex lines passing through a closed curve form a *vortex tube*. Its *strength* is the circulation Γ evaluated on any circuit that links the tube once. Using $\nabla\cdot\boldsymbol{\omega} = 0$ and Gauss' theorem over a volume V bounded by a section of a tube,

$$\int_V \nabla\cdot\boldsymbol{\omega}\,dV = \int_{\partial V}\boldsymbol{\omega}\cdot\mathbf{n}\,dA = \Gamma_{\text{upper}} - \Gamma_{\text{lower}} = 0,$$

so vortex tubes cannot begin or end within the fluid. They may end on a boundary or form closed loops.

Solid-Body Rotation. In solid-body rotation $S_{ij} = 0$, so the Newtonian viscous stress reduces and Cauchy's equation reduces to Euler's equation. With gravity $-g\mathbf{e}_z$ and $u_r = 0, u_\theta = \frac{1}{2}\omega r$, Euler's equations

$$-\rho\frac{u_\theta^2}{r} = -\frac{\partial p}{\partial r}, \quad 0 = -\frac{\partial p}{\partial z} - \rho g,$$

whose integrals are consistent with

$$p(r,z) - p_0 = \frac{\rho\,\omega^2 r^2}{8} - \rho g z.$$

Thus, constant-pressure surfaces are paraboloids of revolution. Because the flow is rotational, the Bernoulli function $B = \frac{1}{2}u_\theta^2 + gz + p/\rho$ is not constant from one streamline to another.

Irrotational Line Vortex. For $u_\theta = \Gamma/(2\pi r)$, fluid elements deform and the shear $\tau_{r\theta} \neq 0$. However, the *net* viscous force on a fluid element vanishes for $r > 0$, so Euler's equations yield

$$p(r,z) - p_\infty = -\frac{\rho\,\Gamma^2}{8\pi^2 r^2} - \rho g z,$$

so constant-pressure surfaces are two-sheeted hyperboloids. Here the Bernoulli relation holds between any two points (steady, incompressible, irrotational flow).

Rotating Cylinder (Rankine Vortex). A solid cylinder of radius a rotating at constant angular rate $\Omega/2$ in a viscous fluid produces the steady field

$$u_\theta(r) = \begin{cases} \frac{1}{2}\,\Omega\, r, & r \le a, \\ \dfrac{\Omega a^2}{2r}, & r \ge a, \end{cases}$$

which is the *Rankine vortex* with core size a and circulation $\Gamma = \pi a^2 \Omega$. Viscous stresses and dissipation are present. The mechanical work at the cylinder wall balances dissipation. The net viscous force at a point is zero in the steady state. By angular momentum balance, the applied torque is transmitted to arbitrarily large radii. In general, viscosity is a primary agent for generating and diffusing vorticity.

6.1 Kelvin's and Helmholtz's Theorems

Helmholtz (1858) established several results on vortex motion in inviscid fluids. A decade later, Kelvin introduced the *circulation.*

Kelvin's Theorem. *In an inviscid, barotropic flow subject to conservative body forces, the circulation around a closed curve that moves with the fluid remains constant in time when observed from a nonrotating/inertial frame.* Equivalently,

$$\frac{D\Gamma}{Dt} = 0,$$

where D/Dt is the material derivative taken following the fluid elements that constitute the closed, material contour C used to define the circulation Γ.

Proof.

$$\frac{D\Gamma}{Dt} = \frac{D}{Dt}\oint_C u_i\,dx_i = \oint_C \frac{Du_i}{Dt}\,dx_i \;+\; \oint_C u_i\,\frac{D}{Dt}(dx_i).$$

Using the momentum equation in component form,

$$\frac{Du_i}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + g_i + \frac{1}{\rho}\frac{\partial \sigma_{ij}}{\partial x_j},$$

$$\oint_C \frac{Du_i}{Dt}\,dx_i = -\oint_C \frac{1}{\rho}\,dp \;-\; \oint_C d\Phi \;+\; \oint_C \frac{1}{\rho}\frac{\partial \sigma_{ij}}{\partial x_j}\,dx_i,$$

where Φ is the body-force potential ($g_i = -\partial\Phi/\partial x_i$). For a barotropic fluid, $\rho = \rho(p)$, and along a closed contour the first two integrals vanish because p, ρ, and Φ are single-valued.

$$u + du = \frac{D}{Dt}(x + dx) = \frac{Dx}{Dt} + \frac{D}{Dt}(dx) \quad\Rightarrow\quad du_i = \frac{D}{Dt}(dx_i),$$

$$\oint_C u_i\,\frac{D}{Dt}(dx_i) = \oint_C u_i\,du_i = \oint_C d\big(\tfrac{1}{2}u_i u_i\big) = 0,$$

again because C is closed and u is single-valued. So,

$$\frac{D\Gamma}{Dt} = \oint_C \frac{1}{\rho}\frac{\partial \sigma_{ij}}{\partial x_j}\,dx_i.$$

Thus, Kelvin's Theorem holds for inviscid flow ($\mu = 0$) or whenever the integrated viscous term vanishes.

Implications for vorticity generation.

(1) Nonzero net viscous torques, typical near solid boundaries under no-slip, where shear generates vorticity.
(2) Nonbarotropic effects (baroclinicity), where ρ depends on variables beyond p (e.g., temperature or composition), producing misaligned ∇p and $\nabla \rho$ and a net torque.
(3) Nonconservative body forces (e.g., Coriolis acceleration in a rotating frame, often coupled with vortex stretching).

Restrictions for irrotational flow to remain irrotational.

(1) No net viscous forces act along C (e.g., C does not enter boundary layers).
(2) The flow is barotropic (e.g., isentropic, isothermal, or constant-density homogeneous flow).
(3) Body forces are conservative, act through the particle center of mass and produce no net torque.
(4) The reference frame is inertial, no extra apparent-force terms from rotation/acceleration.

Helmholtz's Vortex Theorems (under the same restrictions).

(1) Vortex lines move with the fluid.
(2) The strength/circulation of a vortex tube is constant along its length.
(3) A vortex tube cannot end in the interior of the fluid. It must terminate at a boundary or form a closed loop (vortex ring).
(4) The strength of a vortex tube remains constant in time.

6.2 Vorticity Equation in an Inertial Frame of Reference

We derive the vorticity equation for a barotropic fluid of constant density ρ and constant viscosity ν, in an inertial frame. Vorticity is

$$\boldsymbol{\omega} \;\equiv\; \nabla \times \mathbf{u}, \quad \text{so that} \quad \nabla \cdot \boldsymbol{\omega} = 0.$$

Taking the curl of the momentum equation

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho}\nabla p + \mathbf{g} + \nu \nabla^2 \mathbf{u},$$

and for a conservative body force $\mathbf{g} = \nabla\Phi$ we have $\nabla \times \nabla p = \mathbf{0}$ and $\nabla \times \mathbf{g} = \mathbf{0}$, gives

$$\nabla \times \left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) \;=\; \nu\,\nabla \times \nabla^2 \mathbf{u} \;=\; \nu\,\nabla^2(\nabla \times \mathbf{u}) \;=\; \nu\,\nabla^2 \boldsymbol{\omega}.$$

Using the vector identity

$$\nabla \times \big[(\mathbf{u}\cdot\nabla)\mathbf{u}\big] = (\mathbf{u}\cdot\nabla)\boldsymbol{\omega} - (\boldsymbol{\omega}\cdot\nabla)\mathbf{u} \quad (\text{with } \nabla\cdot\mathbf{u} = 0),$$

we obtain

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\mathbf{u}\cdot\nabla)\boldsymbol{\omega} = (\boldsymbol{\omega}\cdot\nabla)\mathbf{u} + \nu\,\nabla^2\boldsymbol{\omega},$$

in material–derivative form,

$$\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega}\cdot\nabla)\mathbf{u} + \nu\,\nabla^2\boldsymbol{\omega}.$$

The diffusion term $\nu\nabla^2\boldsymbol{\omega}$ represents viscous diffusion of vorticity, while $(\boldsymbol{\omega}\cdot\nabla)\mathbf{u}$ is the stretching/tilting of vortex lines. Pressure and conservative body forces do not appear because they exert no net torque on a fluid element. They act through its center of mass.

6.3 Velocity Induced by a Vortex Filament. Law of Biot–Savart

For a variety of applications in aero- and hydrodynamics, one often needs the flow induced by a concentrated distribution of vorticity (a vortex) of arbitrary orientation. Consider incompressible flow with $\nabla\cdot\mathbf{u} = 0$. Taking the curl of the vorticity gives

$$\nabla \times \boldsymbol{\omega} = \nabla \times (\nabla \times \mathbf{u}) = \nabla(\nabla\cdot\mathbf{u}) - \nabla^2\mathbf{u} = -\nabla^2\mathbf{u},$$

so $\mathbf{u}$ satisfies a vector Poisson equation whose solution (the vorticity-induced part of the velocity) is

$$\mathbf{u}(\mathbf{x}, t) = -\frac{1}{4\pi}\int_{V'} \frac{1}{|\mathbf{x}-\mathbf{x}'|}\left(\nabla' \times \boldsymbol{\omega}(\mathbf{x}', t)\right) d^3\mathbf{x}'$$

$$= -\frac{1}{4\pi}\int_{V'} \nabla' \times \left(\frac{\boldsymbol{\omega}(\mathbf{x}', t)}{|\mathbf{x}-\mathbf{x}'|}\right) d^3\mathbf{x}' + \frac{1}{4\pi}\int_{V'} \boldsymbol{\omega}(\mathbf{x}', t) \times \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}\, d^3\mathbf{x}',$$

where V' encloses the vorticity of interest and ∇' acts on $\mathbf{x}'$. If V' is chosen to capture a local segment of the vortex with end faces normal to $\boldsymbol{\omega}$ and lateral surface outside the vorticity support,

$$\mathbf{u}(\mathbf{x}, t) = \frac{1}{4\pi}\int_{V'} \boldsymbol{\omega}(\mathbf{x}', t) \times \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}\, d^3\mathbf{x}'.$$

For a slender vortex element of length dl and cross-sectional area $\Delta A'$, and for observation points sufficiently far that $(\mathbf{x}-\mathbf{x}')/|\mathbf{x}-\mathbf{x}'|^3$ is effectively uniform over the cross-section,

$$d\mathbf{u}(\mathbf{x}, t) \;\approx\; \frac{\Gamma\, dl}{4\pi}\,\hat{\boldsymbol{\omega}} \times \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3},$$

where $\Gamma = \int_{\Delta A'} \boldsymbol{\omega}\cdot\hat{\boldsymbol{\omega}}\, dA'$ is the vortex strength (circulation) and $\hat{\boldsymbol{\omega}}$ is the unit vector along the local vorticity direction. Integrating along a slender filament gives the Biot–Savart law for vortex-induced velocity.

$$\mathbf{u}(\mathbf{x}, t) = \frac{\Gamma}{4\pi}\int_{\text{vortex}} \hat{\boldsymbol{\omega}}(\mathbf{x}') \times \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}\, dl$$

6.4 Vorticity Equation in a Rotating Frame of Reference

The vorticity equation derived in an inertial frame for a uniform–density, uniform–viscosity fluid can be generalized to a steadily rotating frame and to variable density while retaining incompressible flow. This form is relevant to rotating machinery as well as large–scale oceanic and atmospheric motions where Earth's rotation must be included when conserving momentum.

For an incompressible, variable–density flow observed in a frame rotating with constant angular velocity $\boldsymbol{\Omega}$, the continuity and momentum equations are

$$\frac{\partial u_i}{\partial x_i} = 0,$$

$$\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j} + 2\varepsilon_{ijk}\,\Omega_j u_k = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + g_i + \nu\frac{\partial^2 u_i}{\partial x_j \partial x_j},$$

where $\mathbf{g}$ is the effective gravity, including centrifugal effects. The advective and viscous terms can be rewritten to

$$u_j\frac{\partial u_i}{\partial x_j} = -\varepsilon_{ijk}\, u_j\omega_k + \frac{\partial}{\partial x_i}\left(\tfrac{1}{2}u_j u_j\right),$$

$$\nu\frac{\partial^2 u_i}{\partial x_j \partial x_j} = -\nu\,\varepsilon_{ijk}\frac{\partial \omega_k}{\partial x_j},$$

and the Coriolis term may be written as

$$2\varepsilon_{ijk}\,\Omega_j u_k = -2\,\varepsilon_{ijk}\,u_j\Omega_k.$$

The momentum equation becomes

$$\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_i}\left(\tfrac{1}{2}u_j u_j + \Phi\right) - \varepsilon_{ijk}\,u_j(\omega_k + 2\Omega_k) = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} - \nu\,\varepsilon_{ijk}\frac{\partial \omega_k}{\partial x_j}.$$

Taking the Curl, apply $\varepsilon_{nqi}\,\partial/\partial x_q$,

$$\frac{\partial \omega_n}{\partial t} = \frac{\partial u_n}{\partial x_j}\left(\omega_j + 2\Omega_j\right) - u_j\frac{\partial \omega_n}{\partial x_j} + \frac{1}{\rho^2}\,\varepsilon_{nqi}\,\frac{\partial \rho}{\partial x_q}\frac{\partial p}{\partial x_i} + \nu\frac{\partial^2 \omega_n}{\partial x_j \partial x_j}.$$

In vector form,

$$\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega} + 2\boldsymbol{\Omega})\cdot\nabla\,\mathbf{u} + \frac{1}{\rho^2}\,\nabla\rho \times \nabla p + \nu\,\nabla^2\boldsymbol{\omega}$$

which is the variable–density, incompressible vorticity equation in a frame rotating with constant $\boldsymbol{\Omega}$. Here $\mathbf{u}$ is the velocity and $\boldsymbol{\omega} = \nabla\times\mathbf{u}$ is the vorticity measured in the rotating frame. The quantity $2\boldsymbol{\Omega}$ is the *planetary vorticity*, and $\boldsymbol{\omega}_a = \boldsymbol{\omega} + 2\boldsymbol{\Omega}$ is the absolute vorticity. The three right–hand–side terms represent (i) vortex stretching/tilting, (ii) baroclinic generation of vorticity (vanishes for barotropic flows with $\rho = \rho(p)$ s.t. $\nabla\rho \parallel \nabla p$), and (iii) viscous diffusion of vorticity. Introduce the natural orthonormal triad $(\mathbf{e}_s, \mathbf{e}_n, \mathbf{e}_m)$ aligned with the local vorticity direction $\mathbf{e}_s$, arc length s along a vortex line. Then,

$$(\boldsymbol{\omega}\cdot\nabla)\,\boldsymbol{u} = (\boldsymbol{\omega}\cdot\mathbf{e}_s\,\partial/\partial s)\,\boldsymbol{u} = \omega\,\frac{\partial \boldsymbol{u}}{\partial s},$$

so that the s–component governs stretching of vortex lines, whereas the n– and m–components describe tilting.

$$\frac{D\omega_s}{Dt} = \omega\,\frac{\partial u_s}{\partial s}, \quad \frac{D\omega_n}{Dt} = \omega\,\frac{\partial u_n}{\partial s}, \quad \frac{D\omega_m}{Dt} = \omega\,\frac{\partial u_m}{\partial s}.$$

In strictly two–dimensional flows, $(\boldsymbol{\omega}\cdot\nabla)\boldsymbol{u} = \mathbf{0}$.
If $\boldsymbol{\Omega} = \Omega\,\mathbf{e}_z$ and we isolate the rotation–induced production, then

$$\frac{D\omega_z}{Dt} = 2\Omega\,\frac{\partial w}{\partial z}, \quad \frac{D\omega_x}{Dt} = 2\Omega\,\frac{\partial u}{\partial z}, \quad \frac{D\omega_y}{Dt} = 2\Omega\,\frac{\partial v}{\partial z},$$

showing that vertical stretching of fluid columns tends to create vertical relative vorticity.

Kelvin's Theorem with Planetary Vorticity. Inviscid circulation following a material loop in a rotating frame is conserved for the *absolute* vorticity.

$$\frac{D\Gamma_a}{Dt} = 0, \quad \Gamma_a \;\equiv\; \iint_A (\boldsymbol{\omega} + 2\boldsymbol{\Omega})\cdot\mathbf{n}\,dA \;=\; \Gamma \;+\; 2\iint_A \boldsymbol{\Omega}\cdot\mathbf{n}\,dA.$$

7 Ideal Flow

When a constant-density fluid flows without rotation and pressure is measured relative to the local hydrostatic value, the equations of motion in an inertial frame becomes

$$\nabla\cdot\mathbf{u} = 0, \quad \rho\,\frac{D\mathbf{u}}{Dt} = -\nabla p.$$

These are the equations of *ideal flow*. Here, the characteristic size L and speed U are such that the Reynolds number $\mathrm{Re} = \rho UL/\mu$ is large, typically $\mathrm{Re} \gtrsim 10^3$, confining the influence of viscosity and fluid-element rotation to thin surface boundary layers.

7.1 Two-Dimensional Stream Function and Velocity Potential

The two-dimensional incompressible continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

is identically satisfied when the (u, v) velocity components are determined from a single scalar function ψ.

$$u \equiv \frac{\partial \psi}{\partial y}, \quad v \equiv -\frac{\partial \psi}{\partial x}.$$

The function $\psi(x, y)$ is the *stream function* in two dimensions. Along a curve of constant ψ, $d\psi = 0$, which implies

$$0 = d\psi = \frac{\partial \psi}{\partial x}\,dx + \frac{\partial \psi}{\partial y}\,dy = -\,v\,dx + u\,dy, \quad \Rightarrow \quad \left.\frac{dy}{dx}\right|_{\psi=\text{const}} = \frac{v}{u},$$

which is the definition of a streamline in two dimensions. The vorticity ω_z in a flow described by ψ is

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \omega_z = \frac{\partial}{\partial x}\left(-\frac{\partial \psi}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial y}\right) = -\,\nabla^2\psi.$$

In constant-density irrotational flow, ω_z is zero everywhere except at the locations of ideal irrotational vortices. Thus we are interested in solutions of

$$\nabla^2\psi = 0, \quad \nabla^2\psi = -\,\Gamma\,\delta(x - x_0)\,\delta(y - y_0),$$

where δ is the Dirac delta function and $x_0 = (x_0, y_0)$ is the location of an ideal irrotational vortex of strength Γ. In an unbounded domain, the most elementary nontrivial solutions are

$$\psi = -V\,x + U\,y, \quad \psi = -\frac{\Gamma}{2\pi}\,\ln\sqrt{(x - x_0)^2 + (y - y_0)^2},$$

corresponding, respectively, to uniform velocity with horizontal component U and vertical component V, and to the flow induced by an irrotational vortex located at x_0.

An equivalent formulation of two-dimensional ideal flow results when irrotationality is enforced first.

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0,$$

and it is identically satisfied when u, v are obtained from a single scalar function ϕ:

$$u \equiv \frac{\partial \phi}{\partial x}, \quad v \equiv \frac{\partial \phi}{\partial y}.$$

The function $\phi(x, y)$ is the *velocity potential* in two dimensions because it implies $\nabla\phi = \mathbf{u}$. Curves of $\phi = \text{const}$ satisfy

$$0 = d\phi = \frac{\partial \phi}{\partial x}\,dx + \frac{\partial \phi}{\partial y}\,dy = u\,dx + v\,dy, \quad \Rightarrow \quad \left.\frac{dy}{dx}\right|_{\phi=\text{const}} = -\,\frac{u}{v},$$

and are perpendicular to streamlines. Using $\phi(x, y)$, the condition for incompressibility becomes

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial y}\right) = \nabla^2\phi = q(x, y),$$

where $q(x, y)$ is the spatial distribution of source strength in the flow field. In real incompressible flows, $q \equiv 0$. However, ideal point sources and sinks are useful idealizations. They are the ϕ-field counterparts of positive- and negative-circulation ideal vortices in ψ-fields. Thus we are interested in solutions of

$$\nabla^2\phi = 0, \quad \nabla^2\phi = q_s\,\delta(x - x_0)\,\delta(y - y_0),$$

where q_s (units of length^2/time) sets the strength of the singularity at x_0. In an unbounded domain, the most elementary solutions are

$$\phi = U\,x + V\,y, \quad \phi = \frac{q_s}{2\pi}\,\ln\sqrt{(x - x_0)^2 + (y - y_0)^2},$$

corresponding, respectively, to uniform flow with components (U, V) and to the flow induced by an ideal point source of strength q_s located at x_0. Here q_s is the source's volume flow rate per unit length perpendicular to the plane of the flow.

Conservation of mass requires the normal component of fluid velocity to equal the boundary's normal velocity, $\mathbf{n}\cdot\mathbf{U}_s = \mathbf{n}\cdot\mathbf{u}$ on the surface, where $\mathbf{n}$ is the outward normal and $\mathbf{U}_s$ is the surface velocity. For a stationary body this reduces to

$$\left.\frac{\partial \phi}{\partial n}\right|_{\text{surface}} = 0 \quad \text{or} \quad \left.\frac{\partial \psi}{\partial s}\right|_{\text{surface}} = 0,$$

where s is arc length along the surface and n the surface-normal coordinate. Because $\partial\psi/\partial s = 0$ along a streamline, a stationary solid boundary in an ideal flow is itself a streamline. Hence, replacing any ideal-flow streamline by a stationary solid boundary of the same shape leaves the rest of the flow unchanged.

The pressure is then obtained from conservation of momentum via a Bernoulli equation.

$$p + \frac{1}{2}\,\rho\,|\mathbf{u}|^2 = p + \frac{1}{2}\,\rho\,(u^2 + v^2) = p + \frac{1}{2}\,\rho\,|\nabla\phi|^2 = p + \frac{1}{2}\,\rho\,|\nabla\psi|^2 = \text{const}.$$

Planar polar coordinates.

$$\frac{1}{r}\,\frac{\partial}{\partial r}(r\,u_r) + \frac{1}{r}\,\frac{\partial u_\theta}{\partial \theta} = 0 \quad \text{(continuity)},$$

$$\frac{1}{r}\,\frac{\partial}{\partial r}(r\,u_\theta) - \frac{1}{r}\,\frac{\partial u_r}{\partial \theta} = 0 \quad \text{(irrotationality)},$$

and

$$u_r = \frac{\partial \phi}{\partial r} = \frac{1}{r}\,\frac{\partial \psi}{\partial \theta}, \quad u_\theta = \frac{1}{r}\,\frac{\partial \phi}{\partial \theta} = -\,\frac{\partial \psi}{\partial r}.$$

7.2 Construction of Elementary Flows in Two Dimensions

Quadratic functions of elementary flows in x and y are

$$\psi = 2Axy \quad \text{or} \quad \phi = 2Axy,$$

$$\psi = A\,(x^2 - y^2) \quad \text{or} \quad \phi = A\,(x^2 - y^2),$$

where A is a constant. Curves of

$$\psi = -\frac{\Gamma}{2\pi}\,\ln\sqrt{x^2 + y^2} = \text{const}$$

are circles centered at the origin.

$$u = -\frac{\Gamma}{2\pi}\,\frac{y}{x^2 + y^2} = -\frac{\Gamma}{2\pi r}\,\sin\theta, \quad v = \frac{\Gamma}{2\pi}\,\frac{x}{x^2 + y^2} = \frac{\Gamma}{2\pi r}\,\cos\theta.$$

Equivalently, $u_r = 0$ and $u_\theta = \Gamma/(2\pi r)$, i.e. the ideal irrotational vortex. Similarly, curves of

$$\phi = \frac{q_s}{2\pi}\,\ln\sqrt{x^2 + y^2} = \text{const}$$

are circles centered on the origin.

$$u = \frac{q_s}{2\pi}\,\frac{x}{x^2+y^2} = \frac{q_s}{2\pi r}\cos\theta, \quad v = \frac{q_s}{2\pi}\,\frac{y}{x^2+y^2} = \frac{q_s}{2\pi r}\sin\theta.$$

Equivalently, $u_r = q_s/(2\pi r)$ and $u_\theta = 0$, which is radial flow away from the origin. Here, $\nabla\cdot\mathbf{u}$ is zero everywhere except at $r = 0$. Thus, this potential represents flow from an ideal incompressible point source ($q_s > 0$) or sink ($q_s < 0$) located at $r = 0$ in two dimensions.

A source of strength $+q_s$ at $(-\varepsilon, 0)$ and a sink of strength $-q_s$ at $(+\varepsilon, 0)$ can be combined to obtain the potential for a doublet in the limit $\varepsilon \to 0$ and $q_s \to \infty$ such that the dipole strength vector

$$\mathbf{d} = \sum_{\text{sources}} \mathbf{x}_i q_{s,i} = -\varepsilon\,\mathbf{e}_x\,q_s + \varepsilon\,\mathbf{e}_x(-q_s) = -2q_s\varepsilon\,\mathbf{e}_x$$

remains constant (pointing from sink toward source). Using $r^2 = x^2 + y^2$ and expanding the logarithms,

$$\phi \;\longrightarrow\; \frac{q_s\varepsilon}{\pi}\,\frac{x}{r^2} = -\frac{\mathbf{d}\cdot\mathbf{x}}{2\pi r^2} = \frac{\|\mathbf{d}\|}{2\pi}\,\frac{\cos\theta}{r}.$$

Source + uniform stream (half-body).

$$\phi = Ux + \frac{q_s}{2\pi}\ln\sqrt{x^2+y^2} = Ur\cos\theta + \frac{q_s}{2\pi}\ln r,$$
$$\psi = Uy + \frac{q_s}{2\pi}\tan^{-1}\left(\frac{y}{x}\right) = Ur\sin\theta + \frac{q_s}{2\pi}\theta.$$

$$u = U + \frac{q_s}{2\pi}\,\frac{x}{x^2+y^2}, \quad v = \frac{q_s}{2\pi}\,\frac{y}{x^2+y^2}.$$

The stagnation point is at $x = a = q_s/(2\pi U)$, $y = 0$, and the stagnation streamline has $\psi = q_s/2$. The stagnation streamlines form a semi-infinite half-body. The half-width h of the body is

$$h = \frac{q_s}{2\pi U}\,(\pi - \theta), \quad h_{\max} \xrightarrow[\theta\to 0]{} \frac{q_s}{2U}.$$

The pressure coefficient on the surface is

$$C_p \equiv \frac{p - p_\infty}{\frac{1}{2}\rho U^2} = 1 - \frac{\|\mathbf{u}\|^2}{U^2}.$$

Uniform stream + doublet (circular cylinder without circulation). Superpose a horizontal free stream U with a doublet of strength $\mathbf{d} = 2\pi U a^2\,\mathbf{e}_x$.

$$\phi = Ux + Ua^2\frac{x}{x^2+y^2} = U\left(r + \frac{a^2}{r}\right)\cos\theta,$$
$$\psi = Uy - Ua^2\frac{y}{x^2+y^2} = U\left(r - \frac{a^2}{r}\right)\sin\theta.$$

The streamline $\psi = 0$ corresponds to $r = a$ for all θ, i.e. a circular cylinder of radius a. The velocity field is

$$u_r = U\left(1 - \frac{a^2}{r^2}\right)\cos\theta, \quad u_\theta = -\,U\left(1 + \frac{a^2}{r^2}\right)\sin\theta.$$

On $r = a$, $u_r = 0$ and $u_\theta = -2U\sin\theta$, so the cylinder-surface pressure coefficient is

$$C_p(r = a, \theta) = 1 - 4\sin^2\theta.$$

There are stagnation points at $(r, \theta) = (a, 0)$ and (a, π). The fore–aft symmetry of C_p implies no net pressure drag (d'Alembert's paradox).

Adding circulation (lift on a cylinder). Add a point vortex of circulation $-\Gamma$ at the origin.

$$\psi = U\left(r - \frac{a^2}{r}\right)\sin\theta \;+\; \frac{\Gamma}{2\pi}\ln\left(\frac{r}{a}\right).$$

The tangential velocity in the field is $u_\theta = -U(1 + a^2/r^2)\sin\theta - \Gamma/(2\pi r)$, so at the surface,

$$u_\theta(r = a, \theta) = -\,2U\sin\theta - \frac{\Gamma}{2\pi a}.$$

Stagnation points on the surface satisfy

$$\sin\theta = -\,\frac{\Gamma}{4\pi a U}.$$

For $\Gamma < 4\pi a U$, there are two surface stagnation points that move with increasing Γ and coalesce at $\Gamma = 4\pi a U$. For larger Γ, a stagnation point appears off the surface along the negative y-axis at

$$r = \frac{1}{4\pi U}\left(\Gamma \pm \sqrt{\Gamma^2 - (4\pi a U)^2}\right),$$

with the physically relevant root $r > a$. The surface pressure from Bernoulli with $p_\infty + \frac{1}{2}\rho U^2$ as the constant is

$$p(r = a, \theta) = p_\infty + \frac{1}{2}\rho\left[U^2 - \left(2U\sin\theta + \frac{\Gamma}{2\pi a}\right)^2\right].$$

The vertical force per unit span (lift) obtained by integrating pressure around the surface is

$$L = \rho U \Gamma,$$

the Kutta–Zhukhovsky lift theorem.

Method of images (walls). Superposition also allows boundaries to be built in via images. If the unbounded-domain solution satisfies $\nabla^2\psi_1 = -\omega_1(x, y)$, then

$$\nabla^2\psi_2 = -\omega_1(x, y) + \omega_1(x, -y)$$

yields the solution for the same vorticity distribution with a solid wall along the x-axis, with

$$\psi_2 = \psi_1(x, y) - \psi_1(x, -y),$$

so that the zero streamline $\psi_2 = 0$ lies on $y = 0$. Similarly, if $\nabla^2\phi_1 = q_1(x, y)$ in an unbounded domain, then

$$\nabla^2\phi_2 = q_1(x, y) + q_1(x, -y), \quad \phi_2 = \phi_1(x, y) + \phi_1(x, -y),$$

enforces $v = \partial\phi_2/\partial y = 0$ on $y = 0$.

7.3 Complex Potential

Using complex variables, the velocity potential ϕ and stream function ψ introduced in the previous sections can be combined into a single complex function $w(z)$, called the *complex potential.*

$$w(z) = \phi(x, y) + i\,\psi(x, y), \quad z \equiv x + iy = re^{i\theta}.$$

The complex function $w(z)$ is assumed to be analytic, so that its derivative $\mathrm{d}w/\mathrm{d}z$ exists and has the same value regardless of the direction of approach in the complex z-plane. This analyticity requirement leads to the Cauchy–Riemann equations.

$$\frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y}, \quad \frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x}.$$

If ϕ is interpreted as the velocity potential and ψ as the stream function, then w is the complex potential of the flow. In this setting, the complex velocity is given by

$$\frac{\mathrm{d}w}{\mathrm{d}z} = u - iv.$$

Consider first the complex potential corresponding to flow near a corner. For a corner of angle $\alpha = \pi/n$, a suitable complex potential

$$w(z) = Az^n = A\big(re^{i\theta}\big)^n = Ar^n\big(\cos n\theta + i \sin n\theta\big), \quad n \geq \tfrac{1}{2},$$

where A is a real constant. For $n = 2$, the streamlines $\psi = \Im\{w\} = Ar^2 \sin^2 \theta$ describe flow in a region bounded by two perpendicular walls. Extending the field into the second quadrant of the z-plane shows that $n = 2$ also represents flow impinging on a flat wall. The streamlines and equipotential lines are rectangular hyperbolas, and the flow includes a stagnation point, so this configuration is called a stagnation flow. For $n = 1/2$, the streamline pattern corresponds instead to flow past a semi-infinite plate.

$$\frac{\mathrm{d}w}{\mathrm{d}z} = nAz^{n-1} = \frac{A\pi}{\alpha}\, z^{(\pi-\alpha)/\alpha},$$

so that $\mathrm{d}w/\mathrm{d}z = 0$ at $z = 0$ for $\alpha < \pi$, whereas $\mathrm{d}w/\mathrm{d}z \longrightarrow \infty$ at $z = 0$ when $\alpha > \pi$. Thus, the origin is a stagnation point if the corner angle is less than 180°, and a point of unbounded velocity if the angle exceeds 180°. In either case it is a singular point of the flow.

The complex potential for an irrotational vortex of circulation (strength) Γ located at (x_0, y_0) is

$$w(z) = -\frac{i\Gamma}{2\pi} \ln\big(z - z_0\big) = \frac{\Gamma}{2\pi}\,\theta_0 - i\,\frac{\Gamma}{2\pi}\,\ln r_0,$$

where $z_0 = x_0 + iy_0$ is the complex coordinate of the vortex center,

$$r_0 = \sqrt{(x - x_0)^2 + (y - y_0)^2}, \quad \theta_0 = \tan^{-1}\left(\frac{y - y_0}{x - x_0}\right).$$

The complex potential for a source or sink of volumetric flow rate per unit depth q_s, located at (x_0, y_0) is

$$w(z) = \frac{q_s}{2\pi} \ln\big(z - z_0\big) = \frac{q_s}{2\pi} \ln r_0 + i\,\frac{q_s}{2\pi}\,\theta_0.$$

The complex potential for a two-dimensional doublet (dipole) with strength d aligned with the x-axis and located at (x_0, y_0) is

$$w(z) = \frac{d}{2\pi\,(z - z_0)}.$$

7.4 Forces on a Two-Dimensional Body

Blasius Theorem. Consider a stationary object of this type with extent B perpendicular to the plane of the flow, and let D (drag) be the streamwise (x) force component and L (lift) be the cross-stream or lateral (y) force (per unit depth) exerted on the object by the surrounding fluid. Thus, from Newton's third law, the total force applied to the fluid by the object is

$$\boldsymbol{F} = B\big(D\,\boldsymbol{e}_x + L\,\boldsymbol{e}_y\big).$$

For steady, irrotational, constant-density flow, conservation of momentum within a stationary control volume implies

$$\int_{A^*} \rho\,\boldsymbol{u}\,(\boldsymbol{u}\cdot\boldsymbol{n})\,dA \;=\; -\int_{A^*} p\,\boldsymbol{n}\,dA + \boldsymbol{F}.$$

If the control surface A^* is chosen to coincide with the body surface and the body is not moving, then $\boldsymbol{u}\cdot\boldsymbol{n} = 0$, so

$$D\,\boldsymbol{e}_x + L\,\boldsymbol{e}_y \;=\; -\frac{1}{B}\int_{A^*} p\,\boldsymbol{n}\,dA.$$

If C is the contour of the body's cross section, then $dA = B\,ds$ where $ds = \boldsymbol{e}_x\,dx + \boldsymbol{e}_y\,dy$ is an elemental vector along C and $|ds| = \big[(dx)^2 + (dy)^2\big]^{1/2}$.

$$\boldsymbol{n} = \frac{\boldsymbol{e}_x\,dy - \boldsymbol{e}_y\,dx}{|ds|}.$$

$$\begin{aligned} D\,\boldsymbol{e}_x + L\,\boldsymbol{e}_y &= -\frac{1}{B}\oint_C p\,\big(\boldsymbol{e}_x\,dy - \boldsymbol{e}_y\,dx\big)\,\frac{B\,|ds|}{|ds|} \\ &= -\oint_C p\,dy\,\boldsymbol{e}_x + \oint_C p\,dx\,\boldsymbol{e}_y. \end{aligned}$$

$$D - iL = -\oint_C p\,dy - i\oint_C p\,dx = -i\oint_C p\,(dx - i\,dy) = -i\oint_C p\,dz^*,$$

The pressure is found from the Bernoulli equation,

$$p_\infty + \frac{1}{2}\rho U^2 = p + \frac{1}{2}\rho\big(u^2 + v^2\big) = p + \frac{1}{2}\rho\,(u - iv)(u + iv),$$

where p_∞ and U are the pressure and horizontal flow speed far from the body.

$$D - iL = -i\oint_C\Big[p_\infty + \frac{1}{2}\rho U^2 - \frac{1}{2}\rho\,(u - iv)(u + iv)\Big]dz^*.$$

The integral of the constant terms $p_\infty + \frac{1}{2}\rho U^2$ around a closed contour is zero. On the body surface, the velocity vector and the surface element $dz = |dz|\mathrm{e}^{i\alpha}$ are parallel, so

$$(u + iv)\,dz^* = (u - iv)\,dz = \frac{dw}{dz}\,dz,$$

$$D - iL = i\,\frac{\rho}{2}\oint_C\left(\frac{dw}{dz}\right)^2 dz,$$

It applies to any steady planar ideal flow.

Kutta–Zhukhovsky Lift Theorem. The Blasius theorem can be readily applied to an arbitrary cross-section object around which there is circulation Γ. The flow can be considered as a superposition of a uniform stream and a set of singularities such as vortices, doublets, sources, and sinks.

As there are no singularities outside the body, we shall take the contour C in the Blasius theorem at a very large distance from the body. From large distances, all singularities appear to be located near the origin $z = 0$, so the complex potential on the contour C will be of the form

$$w(z) = Uz + \frac{q_s}{2\pi}\ln z + i\,\frac{\Gamma}{2\pi}\ln z + \frac{d}{2\pi z} + \cdots,$$

where U, q_s, Γ, and d are positive and real. The first term represents a uniform flow in the x-direction, the second term represents a net source of fluid, the third term represents a clockwise vortex, and the fourth term represents a doublet. Because the body contour is closed, there can be no net flux of fluid into the domain. Sp, $q_s = 0$.

The Blasius theorem then becomes

$$D - iL = i\,\frac{\rho}{2}\oint_C \left(U + \frac{i\Gamma}{2\pi z} - \frac{d}{2\pi z^2} + \cdots\right)^2 dz$$
$$= i\,\frac{\rho}{2}\oint_C \left[U^2 + \frac{iU\Gamma}{\pi}\,\frac{1}{z} + \left(\frac{Ud}{\pi} - \frac{\Gamma^2}{4\pi^2}\right)\frac{1}{z^2} + \cdots\right] dz.$$
$$D - iL = i\,\frac{\rho}{2}\,2\pi i\left(\frac{iU\Gamma}{\pi}\right) = -\,i\,\rho U\Gamma,$$

or

$$D = 0, \quad L = \rho U\Gamma.$$

Thus, there is no drag on an arbitrary-cross-section object in steady two-dimensional, irrotational, constant-density flow, a more general statement of d'Alembert's paradox.

8 Gravity Waves

There are three types of waves commonly considered in the study of fluid mechanics: interface waves, internal waves, and compression and expansion waves. For interface waves, the restoring forces are gravity and surface tension. For internal waves, the restoring force is gravity. For expansion and compression waves, the restoring force comes directly from the compressibility of the fluid. Perhaps the simplest and most readily observed fluid waves are those that form and travel on the density discontinuity provided by an air–water interface. Such surface capillary–gravity waves, sometimes simply called water waves, involve fluid particle motions parallel and perpendicular to the direction of wave propagation. Thus, the waves are neither longitudinal nor transverse. Wave amplitudes are assumed small enough so that the governing equations and boundary conditions are linear. For such linear waves, Fourier superposition of sinusoidal waves allows arbitrary waveforms to be constructed and sinusoidal waveforms arise naturally from the linearized equations for water waves. Consequently, a simple sinusoidal traveling wave of the form

$$\eta(x,t) = a\cos\left[\frac{2\pi}{\lambda}\,(x - ct)\right]$$

is a foundational element for what follows. In Cartesian coordinates with x horizontal and z vertical, $z = \eta(x,t)$ specifies the waveform or surface shape where a is the wave amplitude, λ is the wavelength, c is the phase speed, and $2\pi(x - ct)/\lambda$ is the phase. In addition, the spatial frequency $k \equiv 2\pi/\lambda$, with units of rad/m, is known as the wave number. If it describes the vertical deflection of an air–water interface, then the height of wave crests is $+a$ and the depth of the wave troughs is $-a$ compared to the undisturbed water-surface location $z = 0$. At any instant in time, the distance between successive wave crests is λ. At any fixed x-location, the time between passage of successive wave crests is the period, $T = 2\pi/(kc) = \lambda/c$. Thus, the wave's cyclic frequency is $\nu = 1/T$ with units of Hz, and its radian frequency is $\omega = 2\pi\nu$ with units of rad/s.

$$\eta(x,t) = a\cos\big(kx - \omega t\big).$$
$$x_{\text{crest}} = \frac{\omega}{k}\,t + \frac{2\pi n}{k}.$$
$$c = \frac{\omega}{k} = \lambda\nu.$$

A useful three-dimensional generalization is

$$\eta = a\cos\big(kx + ly + mz - \omega t\big) = a\cos\big(\boldsymbol{K}\cdot\boldsymbol{x} - \omega t\big),$$

where $\boldsymbol{K} = (k, l, m)$ is a vector, called the *wave number vector*, whose magnitude K is given by

$$K^2 = k^2 + l^2 + m^2.$$
$$\lambda = \frac{2\pi}{K},$$
$$\boldsymbol{c} = \frac{\omega}{K}\,\boldsymbol{e}_K,$$

where $\boldsymbol{e}_K = \boldsymbol{K}/K$. And, $c_x = \omega/k$, $c_y = \omega/l$, and $c_z = \omega/m$ are each larger than the resultant $c = \omega/K$. Any of the three axis-specific phase speeds is sometimes called the *trace velocity* along its associated axis. If sinusoidal waves exist in a fluid moving with uniform speed $\boldsymbol{U}$, then the observed phase speed is $\boldsymbol{c}_0 = \boldsymbol{c} + \boldsymbol{U}$.

$$\omega_0 = \omega + \boldsymbol{U}\cdot\boldsymbol{K},$$

where ω_0 is the observed frequency at a fixed point, and ω is the intrinsic frequency measured by an observer moving with the flow.

8.1 Linear Liquid-Surface Gravity Waves

We develop the properties of small-slope, small-amplitude gravity waves on the free surface of a constant-density liquid layer of uniform depth H. The limitation to waves with small slopes and amplitudes implies $a/\lambda \ll 1$ and $a/H \ll 1$, respectively. These two conditions allow the problem to be linearized. Surface tension is neglected for simplicity. In addition, the air above the liquid is ignored, and the liquid's motion is presumed to be irrotational and entirely caused by the surface waves. Because the liquid's motion is irrotational, we introduce a velocity potential $\phi(x,z,t)$ such that

$$u = \frac{\partial\phi}{\partial x}, \quad w = \frac{\partial\phi}{\partial z},$$

so that the incompressible continuity equation $\partial u/\partial x + \partial w/\partial z = 0$ implies the Laplace equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial z^2} = 0.$$

There are three boundary conditions. At the bottom $z = -H$ we impose zero normal velocity,

$$w = \frac{\partial\phi}{\partial z} = 0 \quad \text{on } z = -H.$$

At the free surface, we apply a kinematic boundary condition that requires the fluid-particle velocity normal to the surface.

$$\big(\mathbf{n}\cdot\mathbf{u}\big)_{z=\eta} = \mathbf{n}\cdot\mathbf{u}_s,$$

where $\mathbf{n}$ is the unit normal to the free surface. This ensures that the liquid elements that define the interface do not separate from the interface, while still allowing motion tangential to the surface. The free surface is given by the level set

$$F(x,z,t) = z - \eta(x,t) = 0,$$

so the upward-pointing unit normal to the surface is

$$\mathbf{n} = \frac{\nabla F}{|\nabla F|} = \frac{-\eta_x\,\mathbf{e}_x + \mathbf{e}_z}{\sqrt{\eta_x^2 + 1}}, \quad \eta_x \equiv \frac{\partial\eta}{\partial x},$$

and the surface velocity can be taken as purely vertical,

$$\mathbf{u}_s = \eta_t\,\mathbf{e}_z, \quad \eta_t \equiv \frac{\partial\eta}{\partial t}.$$

Using $\mathbf{u} = u\,\mathbf{e}_x + w\,\mathbf{e}_z$ gives

$$(\nabla F \cdot \mathbf{u})_{z=\eta} = \nabla F \cdot \mathbf{u}_s,$$

which becomes

$$\Big(-u\,\eta_x + w\Big)_{z=\eta} = \eta_t, \quad \text{or} \quad \left.\frac{\partial \phi}{\partial z}\right|_{z=\eta} = \eta_t + \eta_x \left.\frac{\partial \phi}{\partial x}\right|_{z=\eta}.$$

For small-slope waves, the last term is small compared to the other two, so the kinematic boundary condition can be approximated by

$$\left.\frac{\partial \phi}{\partial z}\right|_{z=\eta} \simeq \eta_t.$$

$$\left.\frac{\partial \phi}{\partial z}\right|_{z=\eta} = \left.\frac{\partial \phi}{\partial z}\right|_{z=0} + \eta \left.\frac{\partial^2 \phi}{\partial z^2}\right|_{z=0} + \cdots \simeq \eta_t.$$

When a/λ is small enough, the most simplified form of the kinematic boundary condition is

$$\left.\frac{\partial \phi}{\partial z}\right|_{z=0} \simeq \eta_t.$$

In addition to the kinematic condition, there is a dynamic condition at the free surface. The pressure just below the surface equals the ambient pressure, with surface tension neglected.

$$p\big|_{z=\eta} = 0,$$

where p is the gauge pressure. For consistency with the small-slope approximation, we linearize the Bernoulli equation by dropping the nonlinear kinetic-energy term $\frac{1}{2}|\nabla\phi|^2$, giving

$$\frac{\partial \phi}{\partial t} + \frac{p}{\rho} + gz \simeq 0,$$

where the Bernoulli constant has been evaluated on the undisturbed surface far from the wave. Evaluating at $z = \eta$ yields

$$\left.\frac{\partial \phi}{\partial t}\right|_{z=\eta} + g\,\eta \simeq 0.$$

Expanding ϕ_t about $z = 0$ and retaining the leading term gives

$$\left.\frac{\partial \phi}{\partial t}\right|_{z=0} \simeq -g\,\eta.$$

For simplicity, consider $\eta(x, 0) = a\cos(kx)$, consistent with the sinusoidal wave

$$\eta(x,t) = a\cos(kx - \omega t).$$

Thus, we seek a solution of the form

$$\phi(x,z,t) = f(z)\,\sin\big(kx - \omega t\big).$$

$$f''(z) - k^2 f(z) = 0,$$

Hence,

$$\phi(x,z,t) = \big(Ae^{kz} + Be^{-kz}\big)\,\sin(kx - \omega t).$$

$$\left.\frac{\partial \phi}{\partial z}\right|_{z=-H} = k\big(Ae^{-kH} - Be^{kH}\big)\,\sin(kx - \omega t) = 0,$$

so

$$B = Ae^{-2kH}.$$

$$\left.\frac{\partial \phi}{\partial z}\right|_{z=0} = k(A - B)\,\sin(kx - \omega t) = \eta_t = a\omega\,\sin(kx - \omega t),$$

so

$$k(A - B) = a\omega.$$

$$A = \frac{a\omega}{k\big(1 - e^{-2kH}\big)}, \quad B = \frac{a\omega e^{-2kH}}{k\big(1 - e^{-2kH}\big)}.$$

$$\phi(x,z,t) = \frac{a\omega}{k}\,\frac{\cosh\big(k(z+H)\big)}{\sinh(kH)}\,\sin(kx - \omega t),$$

$$u = \frac{\partial \phi}{\partial x} = a\omega\,\frac{\cosh\big(k(z+H)\big)}{\sinh(kH)}\,\cos(kx - \omega t),$$

$$w = \frac{\partial \phi}{\partial z} = a\omega\,\frac{\sinh\big(k(z+H)\big)}{\sinh(kH)}\,\sin(kx - \omega t).$$

$$\left.\frac{\partial \phi}{\partial t}\right|_{z=0} = -\frac{a\omega^2}{k}\,\frac{\cosh(kH)}{\sinh(kH)}\,\cos(kx - \omega t) \simeq -g\,\eta = -ga\cos(kx - \omega t),$$

$$\omega = \sqrt{gk\tanh(kH)}, \quad \text{or} \quad T = \sqrt{\frac{2\pi\lambda}{g}\,\coth\!\left(\frac{2\pi H}{\lambda}\right)},$$

where $T = 2\pi/\omega$ is the wave period and $\lambda = 2\pi/k$ is the wavelength. The phase speed c of the surface waves is

$$c = \frac{\omega}{k} = \sqrt{\frac{g}{k}\,\tanh(kH)} = \sqrt{\frac{g\lambda}{2\pi}\,\tanh\!\left(\frac{2\pi H}{\lambda}\right)}.$$

This shows that linear gravity waves on a free surface are generally dispersive. The phase speed depends on wavenumber, with long waves, small k traveling faster.

Consider the time-dependent perturbation pressure

$$p' \equiv p + \rho g z,$$

produced by the surface waves.

$$\begin{aligned} p' &= -\rho\,\frac{\partial \phi}{\partial t} = \rho\,\frac{a\omega^2}{k}\,\frac{\cosh\big(k(z+H)\big)}{\sinh(kH)}\,\cos(kx - \omega t) \\ &= \rho g a\,\frac{\cosh\big(k(z+H)\big)}{\cosh(kH)}\,\cos(kx - \omega t). \end{aligned}$$

Although surface gravity waves transport energy, they do not, in linear theory, produce net transport of fluid parcels. To see this, consider a fluid particle whose path is $\mathbf{x}_p(t) = x_p(t)\,\mathbf{e}_x + z_p(t)\,\mathbf{e}_z$.

$$\frac{dx_p}{dt} = u\big(x_p, z_p, t\big), \quad \frac{dz_p}{dt} = w\big(x_p, z_p, t\big),$$

$$\begin{aligned} \frac{dx_p}{dt} &= a\omega\,\frac{\cosh\big(k(z_p+H)\big)}{\sinh(kH)}\,\cos\big(kx_p - \omega t\big), \\ \frac{dz_p}{dt} &= a\omega\,\frac{\sinh\big(k(z_p+H)\big)}{\sinh(kH)}\,\sin\big(kx_p - \omega t\big). \end{aligned}$$

To be consistent with the small-amplitude approximation, we linearize these equations by writing

$$x_p(t) = x_0 + x(t), \quad z_p(t) = z_0 + z(t),$$

where (x_0, z_0) is the mean position of the particle and $(x(t), z(t))$ is a small excursion.

$$\begin{aligned} \frac{dx}{dt} &\simeq a\omega\,\frac{\cosh\big(k(z_0+H)\big)}{\sinh(kH)}\,\cos\big(kx_0 - \omega t\big), \\ \frac{dz}{dt} &\simeq a\omega\,\frac{\sinh\big(k(z_0+H)\big)}{\sinh(kH)}\,\sin\big(kx_0 - \omega t\big). \end{aligned}$$

Integrating in time gives

$$x(t) \simeq -a \frac{\cosh\big(k(z_0 + H)\big)}{\sinh(kH)} \sin\big(kx_0 - \omega t\big),$$
$$z(t) \simeq a \frac{\sinh\big(k(z_0 + H)\big)}{\sinh(kH)} \cos\big(kx_0 - \omega t\big),$$

which are purely oscillatory. There is no term that grows with t, so the mean position (x_0, z_0) is time-independent to this order.

$$\frac{x^2}{\big[a \cosh\big(k(z_0 + H)\big)/\sinh(kH)\big]^2} + \frac{z^2}{\big[a \sinh\big(k(z_0 + H)\big)/\sinh(kH)\big]^2} = 1,$$

showing that the particle moves on an ellipse. Both semi-axes decrease with depth, and the minor semi-axis vanishes at $z_0 = -H$. The motion of fluid particles in any vertical column is in phase. When one is at the top of its orbit, all particles at that x_0 are at the top of their orbits.

The streamfunction ψ can be obtained from

$$\frac{\partial \psi}{\partial z} = u, \quad -\frac{\partial \psi}{\partial x} = w,$$

$$\psi(x, z, t) = \frac{a\omega}{k} \frac{\sinh\big(k(z + H)\big)}{\sinh(kH)} \cos(kx - \omega t),$$

At any fixed time, the bottom $z = -H$ corresponds to $\psi = 0$, and ψ also vanishes at certain surface locations where $\eta = 0$.

The kinetic energy per unit horizontal area, E_k,

$$E_k = \frac{\rho}{2\lambda} \int_0^\lambda \int_{-H}^0 \left(u^2 + w^2\right) dz\, dx.$$

$$E_k = \frac{1}{2} \rho g \overline{\eta^2},$$

where $\overline{\eta^2}$ is the mean-square surface displacement. The potential energy per unit area, E_p, is the work required to deform an initially flat free surface into the disturbed state.

$$E_p = \frac{\rho g}{\lambda} \int_0^\lambda \int_{-H}^\eta z\, dz\, dx - \frac{\rho g}{\lambda} \int_0^\lambda \int_{-H}^0 z\, dz\, dx$$
$$= \frac{\rho g}{\lambda} \int_0^\lambda \int_0^\eta z\, dz\, dx = \frac{\rho g}{2\lambda} \int_0^\lambda \eta^2\, dx = \frac{1}{2} \rho g \overline{\eta^2}.$$

Thus,

$$E_p = \frac{1}{2} \rho g \overline{\eta^2}, \quad E = E_k + E_p = \rho g \overline{\eta^2}.$$

For a sinusoidal wave with amplitude a, $\overline{\eta^2} = a^2/2$, and

$$E = \frac{1}{2} \rho g a^2,$$

is the total wave energy per unit horizontal area. The time-averaged energy flux E_F across the plane $x = 0$ is the pressure work done by fluid in $x < 0$ on fluid in $x > 0$.

$$E_F = \frac{\omega}{2\pi} \int_0^{2\pi/\omega} \int_{-H}^0 p\, u\, dz\, dt = \frac{\omega}{2\pi} \int_0^{2\pi/\omega} \int_{-H}^0 \big(p' - \rho g z\big)\, u\, dz\, dt.$$
$$= \rho a^2 \frac{\omega^3}{k \sinh^2(kH)} \int_{-H}^0 \cosh^2\big(k(z + H)\big)\, dz$$
$$= \frac{1}{2} \rho g a^2 \left(\frac{c}{2}\right) \left[1 + \frac{2kH}{\sinh(2kH)}\right].$$

Deep-water waves.

$$c = \sqrt{\frac{g}{k} \tanh(kH)}.$$

For $H/\lambda \gg 1$, $kH \gg 1$, $\tanh(kH) \to 1$.

$$c \simeq \sqrt{\frac{g}{k}} = \sqrt{\frac{g\lambda}{2\pi}},$$

Waves satisfying $H > \lambda/3$ are classified as deep-water waves. They are strongly dispersive, since c depends on λ. For $kH \gg 1$,

$$\frac{\cosh\big(k(z + H)\big)}{\sinh(kH)} \simeq \frac{\sinh\big(k(z + H)\big)}{\sinh(kH)} \simeq e^{kz},$$

$$x(t) \simeq -ae^{kz_0} \sin(kx_0 - \omega t), \quad z(t) \simeq ae^{kz_0} \cos(kx_0 - \omega t),$$

describing circular orbits of radius ae^{kz_0}, decreasing with depth. At the surface $z_0 = 0$, the orbit radius is a.

$$u \simeq a\omega e^{kz} \cos(kx - \omega t), \quad w \simeq a\omega e^{kz} \sin(kx - \omega t).$$

At a fixed spatial point, the velocity vector rotates with constant magnitude $a\omega e^{kz}$ and angular frequency ω.

$$p' \simeq \rho g a e^{kz} \cos(kx - \omega t),$$

Wave-induced pressure fluctuations decay exponentially with depth.

Shallow-water waves.

For $H/\lambda \ll 1$, we use $\tanh(x) \simeq x$ for small x. Then,

$$c \simeq \sqrt{gH}.$$

Waves are classified as shallow-water waves only if their wavelength exceeds about $14H$. Shallow-water gravity waves are non-dispersive. c depends on H but not on λ.

$$\cosh\big(k(z + H)\big) \simeq 1, \ \sinh\big(k(z + H)\big) \simeq k(z + H), \ \sinh(kH) \simeq kH,$$

$$x(t) \simeq -\frac{a}{kH} \sin(kx_0 - \omega t), \quad z(t) \simeq a\left(1 + \frac{z_0}{H}\right) \cos(kx_0 - \omega t),$$

which describe thin ellipses whose minor semi-axis decreases linearly to zero at the bottom.

$$u \simeq \frac{a\omega}{kH} \cos(kx - \omega t), \quad w \simeq a\omega\left(1 + \frac{z}{H}\right) \sin(kx - \omega t),$$

so $|w| \ll |u|$ and vertical accelerations are small.

$$p' \simeq \rho g a \cos(kx - \omega t) = \rho g\, \eta,$$

which is independent of depth. The pressure field is purely hydrostatic. The departure from the undisturbed state equals the hydrostatic pressure due to the surface elevation η everywhere in the water column. This is why shallow-water gravity waves are sometimes called hydrostatic waves.

Finally, the depth dependence of the phase speed explains the refraction of long waves approaching coastlines or islands as the depth H decreases. The portion of a wave entering shallower water slows down relative to the part still in deeper water, causing the crest lines to rotate and tend to align with depth contours such as shorelines or circular isobaths around islands. This bending of wave paths in a spatially varying medium is the phenomenon of wave refraction, analogous to refraction of light in a non-uniform refractive index field.

8.2 Standing Waves

If two waves of equal amplitude and wavelength traveling oppositely, their superposition can produce a non-propagating pattern.

$$\eta = a\cos(kx - \omega t) + a\cos(kx + \omega t) = 2a\cos(kx)\cos(\omega t).$$

At locations where $kx = \pm\pi/2,\ \pm 3\pi/2, \ldots$, the surface displacement η is identically zero for all t. These fixed points of zero displacement are called *nodes*. For such a motion the free surface does not carry disturbances downstream. Instead, the surface oscillates vertically with frequency ω, and the oscillation amplitude varies in space while the nodal points remain fixed. This type of motion is called a *standing wave*.

$$\begin{aligned}\psi &= \frac{a\omega}{k}\,\frac{\sinh\big(k(z+H)\big)}{\sinh(kH)}\Big[\cos(kx - \omega t) - \cos(kx + \omega t)\Big] \\ &= \frac{2a\omega}{k}\,\frac{\sinh\big(k(z+H)\big)}{\sinh(kH)}\,\sin(kx)\sin(\omega t).\end{aligned}$$

Standing waves naturally arise in a confined region of water, such as a tank, pool, or lake, when traveling waves reflect from the boundaries. In a lake this type of oscillation is called a *seiche*. Consider an idealized rectangular tank of length L, uniform depth H, and vertical end walls, with the waves assumed independent of y.

$$u = 2a\omega\,\frac{\cosh\big(k(z+H)\big)}{\sinh(kH)}\,\sin(kx)\sin(\omega t).$$

Let the vertical walls be located at $x = 0$ and $x = L$. The no-penetration condition at these walls requires

$$u(x = 0) = 0, \quad u(x = L) = 0.$$

$$\sin(kL) = 0 \quad \Rightarrow \quad kL = n\pi, \quad n = 1, 2, 3, \ldots.$$

Hence the allowed wavelengths are

$$\lambda = \frac{2\pi}{k} = \frac{2L}{n}.$$

$$\omega = \sqrt{\frac{n\pi g}{L}\,\tanh\!\left(\frac{n\pi H}{L}\right)}, \quad n = 1, 2, 3, \ldots.$$

8.3 Group Velocity, Energy Flux, and Dispersion

Wavelength-dependent/dispersive propagation is common for waves that travel on interfaces between different materials. Examples are Rayleigh waves (vacuum and a solid), Stonely waves (a solid and another material), or interface waves (two different immiscible liquids). Here we consider only air–water interface waves and emphasize deep-water gravity waves for which $c \propto \sqrt{\lambda}$. In a dispersive system, the energy of a wave component does not propagate at the phase velocity $c = \omega/k$, but at the group velocity defined as $c_g = d\omega/dk$.

$$\begin{aligned}\eta &= a\cos(k_1 x - \omega_1 t) + a\cos(k_2 x - \omega_2 t) \\ &= 2a\cos\!\left(\frac{1}{2}\Delta k\,x - \frac{1}{2}\Delta\omega\,t\right)\cos(kx - \omega t),\end{aligned}$$

where $\Delta k = k_2 - k_1$ and $\Delta\omega = \omega_2 - \omega_1$, while $k = (k_1 + k_2)/2$ and $\omega = (\omega_1 + \omega_2)/2$. Here, $\cos(kx - \omega t)$ is a progressive wave with a phase speed $c = \omega/k$. However, its amplitude $2a$ is modulated by a slowly varying function $\cos(\Delta k\,x/2 - \Delta\omega\,t/2)$, which has a large wavelength $4\pi/\Delta k$, a long period $4\pi/\Delta\omega$, and propagates at a speed

$$c_g = \frac{\Delta\omega}{\Delta k} \approx \frac{d\omega}{dk},$$

where the approximate equality becomes exact in the limit as Δk and $\Delta\omega \to 0$. Multiplication of a rapidly varying sinusoid and a slowly varying sinusoid, generates repeating wave groups. The individual wave crests and troughs propagate with the speed $c = \omega/k$, but the envelope of the wave groups travels with the speed c_g, which is therefore called the group velocity. If $c_g < c$, then individual wave crests appear spontaneously at a nodal point, proceed forward through the wave group, and disappear at the next nodal point. If, on the other hand, $c_g > c$, then individual wave crests emerge from a forward nodal point and vanish at a backward nodal point. The subsequent evolution of the wave is approximately described by

$$\eta = a\big(x - c_g t\big)\cos(kx - \omega t),$$

where $c_g = d\omega/dk$. This shows that the amplitude of a wave packet travels with the group speed. It follows that c_g must equal the speed of propagation of energy of a certain wavelength. The fact that c_g is the speed of energy propagation is also evident from the behavior of modulated wave trains because the nodal points travel at c_g and no energy crosses nodal points since $p' = 0$ there.

$$c_g = \frac{c}{2}\left(1 + \frac{2kH}{\sinh(2kH)}\right),$$

which has two limiting cases.

$$c_g = \frac{c}{2} \quad \text{(deep water)}, \qquad c_g = c \quad \text{(shallow water)}.$$

The group velocity of deep-water gravity waves is half the deep-water phase speed while shallow-water waves are non-dispersive with $c = c_g$. For a linear non-dispersive system, any waveform preserves its shape as it travels because all the wavelengths that make up the waveform travel at the same speed.

$$E_F = E\,\frac{c}{2}\left(1 + \frac{2kH}{\sinh(2kH)}\right) = Ec_g,$$

where $E = \rho g a^2/2$ is the average energy in the water column per unit horizontal area. This signifies that the rate of transmission of energy of a sinusoidal wave component is the wave energy times the group velocity, and reinforces the interpretation of the group velocity as the speed of propagation of wave energy. In three dimensions, the dispersion relation $\omega = \omega(k, \ell, m)$ may depend on all three components of the wave number vector $\boldsymbol{K} = (k, \ell, m)$.

$$c_{gi} = \frac{\partial\omega}{\partial K_i}.$$

Another way to understand the group velocity is to consider the k or λ determined by an observer traveling at speed c_g with a slowly varying wave train described by

$$\eta = a(x, t)\cos\big[\theta(x, t)\big],$$

For a slowly varying wave train, define the local wave number $k(x, t)$ and the local frequency $\omega(x, t)$ as the rate of change of phase in space and time, respectively.

$$k(x, t) \equiv \frac{\partial\theta}{\partial x}, \ \ \omega(x, t) \equiv -\frac{\partial\theta}{\partial t} \ \Rightarrow \ \frac{\partial k}{\partial t} + \frac{\partial\omega}{\partial x} = 0 \ \Rightarrow \ \frac{\partial k}{\partial t} + c_g\frac{\partial k}{\partial x} = 0.$$

Now consider the same traveling observer, but allow there to be smooth variations in the water depth $H(x)$.

$$\omega = \sqrt{gk\ \tanh\left[kH(x)\right]},$$

$$\frac{\partial \omega}{\partial t} + c_g \frac{\partial \omega}{\partial x} = 0.$$

8.4 Nonlinear Waves in Shallow and Deep Water

Consider a finite-amplitude surface displacement consisting of a wave crest and trough, propagating in shallow water of undisturbed depth H. Let a little wavelet be superposed on the crest at point x', at which the water depth is H_0 and the fluid velocity due to the wave motion is $u(x')$. Relative to an observer moving with the fluid velocity u, the wavelet propagates at the local shallow-water speed $c_0 = \sqrt{gH_0}$. The speed of the wavelet relative to a frame of reference fixed in the undisturbed fluid is therefore $c = c_0 + u$. It is apparent that the local wave speed c is no longer constant because $c_0(x)$ and $u(x)$ are variables. This is in contrast to the linearized theory in which u is negligible and c_0 is constant because $H_0 \approx H$.

Let us now examine the effect of variable phase speed on the wave profile. The value of c_0 is larger for points near the wave crest than for points in the wave trough. It follows that the wave speed c is larger for points on the crest than for points on the trough, so that the waveform deforms as it propagates, the crest region tending to overtake the trough region.

The front face is rising with time, and this implies an increase in pressure at any depth within the liquid. The net effect of nonlinearity is a steepening of the compression region. For finite-amplitude waves in a non-dispersive medium like shallow water, therefore, there is an important distinction between compression and expansion regions. A compression region tends to steepen with time, while an expansion region tends to flatten out. This eventually would lead to a wave shape in which there are three values of surface elevation at a point. This situation is certainly possible for time-evolving waves and is readily observed as plunging breakers develop in the surf zone along ocean coastlines.

To analyze a hydraulic jump, consider the flow in a shallow canal of depth H. If the flow speed is u, we may define a dimensionless speed via the Froude number,

$$\mathrm{Fr} \equiv \frac{u}{\sqrt{gH}} = \frac{u}{c}.$$

The Froude number is analogous to the Mach number in compressible flow. The flow is called *supercritical* if $\mathrm{Fr} > 1$, and *subcritical* if $\mathrm{Fr} < 1$. For the situation where the jump is stationary, the upstream flow is supercritical while the downstream flow is subcritical, just as a compressible flow changes from supersonic to subsonic by going through a shockwave. The depth of flow is greater downstream of a hydraulic jump, just as the gas pressure is greater downstream of a shockwave. However, dissipative processes act within shockwaves and hydraulic jumps so that mechanical energy is converted into thermal energy in both cases. An example of a stationary hydraulic jump is found at the foot of a dam, where the flow almost always reaches a supercritical state because of the free fall. A tidal bore propagating into a river mouth is an example of a propagating hydraulic jump. A circular hydraulic jump can be made by directing a vertically falling water stream onto a flat horizontal surface.

The planar hydraulic jump can be analyzed by using a rectangular control volume, the goal being to determine how the depth ratio depends on the upstream Froude number. Q is the volume flow rate per unit width normal to the plane of the paper, then mass conservation requires

$$Q = u_1 H_1 = u_2 H_2.$$

Conserving momentum with the same control volume produces

$$\rho Q\,(u_2 - u_1) = \frac{1}{2}\,\rho g\,(H_1^2 - H_2^2),$$

where the left-hand term comes from the outlet and inlet momentum fluxes, and the right-hand term is the hydrostatic pressure force.

$$Q^2\left(\frac{1}{H_2} - \frac{1}{H_1}\right) = \frac{1}{2} g\,(H_1^2 - H_2^2).$$

$$\left(\frac{H_2}{H_1}\right)^2 + \frac{H_2}{H_1} - 2\mathrm{Fr}_1^2 = 0,$$

where

$$\mathrm{Fr}_1^2 = \frac{Q^2}{gH_1^3} = \frac{u_1^2}{gH_1}.$$

The physically meaningful solution is

$$\frac{H_2}{H_1} = \frac{1}{2}\left(-1 + \sqrt{1 + 8\mathrm{Fr}_1^2}\right).$$

For supercritical flows $\mathrm{Fr}_1 > 1$, it requires that $H_2 > H_1$, and this verifies that water depth increases through a hydraulic jump.

Though a solution with $H_2 < H_1$ for $\mathrm{Fr}_1 < 1$ is mathematically allowed, such a solution violates the second law of thermodynamics, implying an increase of mechanical energy through the jump. Consider the mechanical energy of a fluid particle at the surface,

$$E = \frac{u^2}{2} + gH = \frac{Q^2}{2H^2} + gH.$$

$$E_2 - E_1 = -(H_2 - H_1)\,\frac{g\,(H_2 - H_1)^2}{4H_1 H_2}.$$

This shows that $H_2 < H_1$ implies $E_2 > E_1$, which violates the second law of thermodynamics. The mechanical energy, in fact, decreases in a hydraulic jump because of the action of viscosity.

In a non-dispersive medium, nonlinear effects may continually accumulate until they become large changes. Such an accumulation is prevented in a dispersive medium because the different Fourier components propagate at different speeds and tend to separate from each other. In a dispersive system, then, nonlinear steepening could cancel out the dispersive spreading, resulting in finite-amplitude waves of constant form.

In 1847 Stokes showed that periodic waves of finite amplitude are possible in deep water. In terms of a power series in the amplitude a, he showed that the surface deflection of irrotational waves in deep water is given by

$$\eta = a\cos[k(x-ct)] + \frac{1}{2}ka^2\cos[2k(x-ct)] + \frac{3}{8}k^2a^3\cos[3k(x-ct)] + \cdots,$$

where the speed of propagation is

$$c = \sqrt{\frac{g}{k}\,(1 + k^2a^2 + \cdots)}.$$

Periodic finite-amplitude irrotational waves in deep water are frequently called *Stokes waves.* They have flattened troughs and peaked crests. The maximum possible amplitude is $a_{\max} = 0.07\lambda$, at which point the crest becomes a sharp 120° angle. Attempts at generating waves of larger amplitude result in the appearance of white foam at these sharp crests.

When finite-amplitude waves are present, fluid particles no longer trace closed orbits, but undergo a slow drift in the direction of wave propagation. This is called *Stokes drift.* The mean velocity of a fluid particle is therefore not zero. The drift occurs because the particle moves forward faster when at the top of its trajectory than it does backward when at the bottom of its trajectory.

The fluid particle trajectory $\mathbf{x}_p(t) = x_p(t)\,\mathbf{e}_x + z_p(t)\,\mathbf{e}_z$,

$$\frac{\mathrm{d}x_p(t)}{\mathrm{d}t} = u(x_p, z_p, t) = u(x_0, z_0, t) + x\left.\frac{\partial u}{\partial x}\right|_{x_0,z_0} + z\left.\frac{\partial u}{\partial z}\right|_{x_0,z_0} + \cdots,$$
$$\frac{\mathrm{d}z_p(t)}{\mathrm{d}t} = w(x_p, z_p, t) = w(x_0, z_0, t) + x\left.\frac{\partial w}{\partial x}\right|_{x_0,z_0} + z\left.\frac{\partial w}{\partial z}\right|_{x_0,z_0} + \cdots,$$

where (x_0, z_0) is the fluid element location absence of wave motion.

For deep-water gravity waves, the Stokes drift speed

$$u_L = a^2\omega k\,\mathrm{e}^{2kz_0}.$$

For arbitrary water depth, it may be generalized to

$$u_L = a^2\omega k\frac{\cosh\left[2k(z_0 + H)\right]}{2\sinh^2(kH)}.$$

The Stokes drift causes mass transport in the fluid so it is also called the *mass transport velocity.*

9 Laminar Flow

For low values of the Reynolds number, the entire flow may be influenced by viscosity, and inviscid flow theory is no longer even approximately correct. Viscous flows generically fall into two categories, laminar and turbulent, but the boundary between them is imperfectly defined. The basic difference between the two categories is phenomenological and was dramatically demonstrated in 1883 by Reynolds, who injected a thin stream of dye into the flow of water through a tube. At low flow rates, the dye stream was observed to follow a well-defined straight path, indicating that the fluid moved in parallel layers with no unsteady macroscopic mixing or overturning motion of the layers. Such smooth orderly flow is called *laminar.* However, if the flow rate was increased beyond a certain critical value, the dye streak broke up into irregular filaments and spread throughout the cross-section of the tube, indicating the presence of unsteady, apparently chaotic three-dimensional macroscopic mixing motions. Such irregular disorderly flow is called *turbulent.* Reynolds demonstrated that the transition from laminar to turbulent flow always occurred at or near a fixed value of the ratio that bears his name, the Reynolds number,

$$\mathrm{Re} = \frac{Ud}{\nu} \approx 2000 \text{ to } 3000,$$

where U is the velocity averaged over the tube's cross-section, d is the tube diameter, and $\nu = \mu/\rho$ is the kinematic viscosity. The fluid's kinematic viscosity specifies the propensity for vorticity to diffuse through a fluid. Since ν has the units of (length)^2/time, the kinematic viscosity ν is sometimes called the *momentum diffusivity.* The velocity boundary conditions on a solid surface are

$$\boldsymbol{n}\cdot\boldsymbol{u}_s = \boldsymbol{n}\cdot\boldsymbol{u} \quad \text{on the surface,}$$
$$\boldsymbol{t}\cdot\boldsymbol{u}_s = \boldsymbol{t}\cdot\boldsymbol{u} \quad \text{on the surface,}$$

where $\boldsymbol{u}_s$ is the velocity of the surface, $\boldsymbol{n}$ is the normal to the surface, and $\boldsymbol{t}$ is the tangent to the surface in the plane of interest. Here, fluid density will be assumed constant, and the frame of reference will be inertial. Thus, gravity can be dropped from the momentum equation as long as no free surface is present.

9.1 Exact Solutions for Steady Incompressible Viscous Flow

9.1.1 Steady Flow between Parallel Plates. Consider a viscous, incompressible fluid flowing between two infinite parallel plates aligned with the x-axis. The lower plate is at $y = 0$ and is stationary, while the upper plate at $y = h$ moves in the x-direction with speed U. A constant pressure gradient $\partial p/\partial x \neq 0$ is imposed in the streamwise direction. We assume there is no variation in the z-direction, so that $w = 0$ and $\partial/\partial z = 0$. For a steady, fully developed flow,

$$\boldsymbol{u} = (u(y),\ 0,\ 0),$$

so that $\partial u/\partial x = 0$. The continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

then reduces to $\partial v/\partial y = 0$, and the boundary conditions $v = 0$ at $y = 0$ and $y = h$ imply $v = 0$ everywhere in the channel. Under these conditions the x- and y-momentum equations become

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\mathrm{d}^2u}{\mathrm{d}y^2},$$
$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial y}.$$

Hence, $p = p(x)$, and the pressure gradient must be a constant. The x-momentum equation can then be written as

$$\mu\frac{\mathrm{d}^2u}{\mathrm{d}y^2} = \frac{\mathrm{d}p}{\mathrm{d}x},$$

where $\mu = \rho\nu$ is the dynamic viscosity.

$$u(y) = \frac{1}{2\mu}\frac{\mathrm{d}p}{\mathrm{d}x}\,y^2 + Ay + B,$$

where A and B are constants of integration. The no-slip boundary conditions $u(0) = 0$ and $u(h) = U$ determine these constants

$$u(y) = U\,\frac{y}{h} - \frac{1}{2\mu}\frac{\mathrm{d}p}{\mathrm{d}x}\,y\,(h - y).$$

Depending on the sign of $\mathrm{d}p/\mathrm{d}x$, it may be concave or convex. The volume flow rate per unit width of the channel is

$$\begin{aligned} q &= \int_0^h u(y)\,\mathrm{d}y \\ &= \int_0^h \left[U\,\frac{y}{h} - \frac{1}{2\mu}\,\frac{\mathrm{d}p}{\mathrm{d}x}\,y\,(h-y) \right] \mathrm{d}y \\ &= \frac{Uh}{2}\left[1 - \frac{h^2}{6\mu U}\,\frac{\mathrm{d}p}{\mathrm{d}x} \right]. \end{aligned}$$

The average velocity across the gap, $V = q/h$, is therefore

$$V = \frac{1}{h}\int_0^h u(y)\,\mathrm{d}y = \frac{U}{2}\left[1 - \frac{h^2}{6\mu U}\,\frac{\mathrm{d}p}{\mathrm{d}x} \right].$$

A negative pressure gradient increases the flow rate, while a positive gradient decreases it.

- **Plane Couette flow.** If the motion is driven solely by the moving upper plate, so that $\mathrm{d}p/\mathrm{d}x = 0$,

  $$u(y) = U\,\frac{y}{h}.$$

  The shear stress is uniform across the channel,

  $$\tau = \mu\frac{\mathrm{d}u}{\mathrm{d}y} = \mu\frac{U}{h}.$$

- **Plane Poiseuille flow.** If both plates are stationary ($U = 0$) and the flow is driven purely by the pressure gradient, then

  $$u(y) = -\frac{1}{2\mu}\,\frac{\mathrm{d}p}{\mathrm{d}x}\,y\,(h-y),$$

  a parabolic distribution. The corresponding shear stress is

  $$\tau = \mu\frac{\mathrm{d}u}{\mathrm{d}y} = -\left(\frac{h}{2} - y\right)\frac{\mathrm{d}p}{\mathrm{d}x}.$$

9.1.2 Steady Flow in a Round Tube. Next consider steady, fully developed laminar flow through a circular tube of radius a. We employ cylindrical coordinates (R, ϕ, z) with the tube axis along z. The only nonzero component of velocity is the axial component $u_z(R)$, so that

$$\boldsymbol{u} = (0,\ 0,\ u_z(R)).$$

This automatically satisfies the continuity equation for incompressible flow. From the radial and azimuthal momentum equations one finds that the pressure does not depend on R or ϕ, and hence $p = p(z)$ is a linear function of z. The z-momentum equation reduces to

$$0 = -\frac{\mathrm{d}p}{\mathrm{d}z} + \mu\frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}R}\left(R\frac{\mathrm{d}u_z}{\mathrm{d}R}\right).$$

Since $\mathrm{d}p/\mathrm{d}z$ is constant, this equation can be integrated twice with respect to R to give

$$u_z(R) = \frac{1}{4\mu}\,\frac{\mathrm{d}p}{\mathrm{d}z}\,R^2 + A\ln R + B.$$

To keep the velocity finite at the axis $R = 0$, we must set $A = 0$. The no-slip condition $u_z(a) = 0$ then yields

$$B = -\frac{1}{4\mu}\,\frac{\mathrm{d}p}{\mathrm{d}z}\,a^2,$$

and hence the velocity profile becomes

$$u_z(R) = \frac{R^2 - a^2}{4\mu}\,\frac{\mathrm{d}p}{\mathrm{d}z},$$

a parabola of maximum magnitude at the centerline $R = 0$.

From Appendix B, the shear stress in cylindrical coordinates is

$$\tau_{zR} = \mu\left(\frac{\partial u_R}{\partial z} + \frac{\partial u_z}{\partial R}\right).$$

Here $u_R = 0$, so

$$\tau \equiv \tau_{zR} = \mu\frac{\mathrm{d}u_z}{\mathrm{d}R} = \frac{R}{2}\,\frac{\mathrm{d}p}{\mathrm{d}z},$$

and we obtain

$$\tau(R) = \frac{R}{2}\,\frac{\mathrm{d}p}{\mathrm{d}z},$$

which varies linearly with radius. Its magnitude at the wall $R = a$ is

$$\tau_w = \frac{a}{2}\,\frac{\mathrm{d}p}{\mathrm{d}z}.$$

The volume flow rate through the tube is

$$\begin{aligned} Q &= \int_0^a u_z(R)\,2\pi R\,\mathrm{d}R \\ &= \int_0^a \frac{R^2 - a^2}{4\mu}\,\frac{\mathrm{d}p}{\mathrm{d}z}\,2\pi R\,\mathrm{d}R \\ &= -\frac{\pi a^4}{8\mu}\,\frac{\mathrm{d}p}{\mathrm{d}z}, \end{aligned}$$

where the minus sign arises because $\mathrm{d}p/\mathrm{d}z < 0$ for flow in the positive z-direction. The average velocity in the tube is

$$V = \frac{Q}{\pi a^2} = -\frac{a^2}{8\mu}\,\frac{\mathrm{d}p}{\mathrm{d}z}.$$

As in the plane channel, the linear axial pressure variation and the stress distribution of the laminar solution also have counterparts in turbulent pipe flow when suitable averages are taken.

9.1.3 Steady Flow between Concentric Rotating Cylinders. As a third example, consider viscous flow in the annular region between two infinitely long, concentric cylinders. The inner cylinder of radius R_1 rotates with angular velocity U_1, and the outer cylinder of radius R_2 rotates with angular velocity U_2. We again use cylindrical coordinates (R, ϕ, z) and assume the flow is purely azimuthal and independent of ϕ and z,

$$\boldsymbol{u} = (0,\ u_\phi(R),\ 0).$$

The continuity equation is automatically satisfied. The radial and azimuthal momentum equations then reduce to

$$\frac{u_\phi^2}{R} = \frac{1}{\rho}\,\frac{\mathrm{d}p}{\mathrm{d}R}, \quad 0 = \mu\frac{\mathrm{d}}{\mathrm{d}R}\left[\frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}R}(Ru_\phi)\right].$$

The first equation states that the radial pressure gradient balances the centrifugal acceleration; the second is the governing equation for the tangential velocity. Integrating the azimuthal equation twice yields

$$u_\phi(R) = AR + \frac{B}{R},$$

where A and B are constants. The no-slip conditions at the cylinder surfaces,

$$u_\phi(R_1) = U_1R_1, \quad u_\phi(R_2) = U_2R_2,$$

determine A and B:

$$A = \frac{U_2 R_2^2 - U_1 R_1^2}{R_2^2 - R_1^2},$$

$$B = \frac{(U_2 - U_1)\, R_1^2 R_2^2}{R_2^2 - R_1^2}.$$

Substitution gives the general velocity distribution

$$u_\phi(R) = \frac{1}{R_2^2 - R_1^2}\left[(U_2 R_2^2 - U_1 R_1^2)\, R - (U_2 - U_1)\, \frac{R_1^2 R_2^2}{R}\right].$$

Two limiting cases are especially instructive:

- **Rotating cylinder in an unbounded fluid.** Let $R_2 \to \infty$ with $U_2 = 0$, so the outer cylinder recedes to infinity and the fluid extends without bound. Simplifying (**??**) gives

  $$u_\phi(R) = \frac{U_1 R_1^2}{R},$$

  which has the form of an ideal (irrotational) vortex for $R > R_1$. Although the flow is irrotational, viscous shear stresses are present. In cylindrical coordinates the relevant component is

  $$\tau_{R\phi} = \mu\left[\frac{1}{R}\frac{\partial u_R}{\partial \phi} + R\frac{\partial}{\partial R}\left(\frac{u_\phi}{R}\right)\right],$$

  and here $u_R = 0$, so

  $$\tau_{R\phi} = 2\mu\, \frac{U_1 R_1^2}{R^2}.$$

  The mechanical power supplied to the fluid (per unit cylinder length) equals the rate of viscous energy dissipation in the flow.

- **Solid-body rotation in a cylindrical tank.** Let $R_1 \to 0$ and $U_1 = 0$ so that the inner cylinder disappears, and the outer cylinder of radius R_2 rotates at angular velocity U_2. Then (**??**) reduces to

  $$u_\phi(R) = U_2 R,$$

  corresponding to solid-body rotation.

In each of the three examples of this section, the velocity field is confined between solid boundaries, and the symmetries of the configuration remove the convective acceleration terms from the governing equations. Many other exact solutions of the steady incompressible Navier–Stokes equations (both internal and external) exist and can be found in specialized references. Before turning to additional examples and the entrance-region development of internal flows, we next introduce a brief treatment of classical lubrication theory, which is another important limiting case of viscous flow in thin geometries.

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References

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