Vector Calculus And Complex Variables

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These notes are primarily based on the textbook by kreyszig [1].

1 Vector Differential Calculus

1.1 Components

$$\mathbf{a} = [a_1, a_2, a_3], \ \ \|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$

1.2 Inner Product (Dot Product)

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \, \|\mathbf{b}\| \cos\gamma = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

$$\text{Cauchy–Schwarz.} \quad |\mathbf{a} \cdot \mathbf{b}| \le \|\mathbf{a}\| \, \|\mathbf{b}\|.$$

1.3 Vector Product (Cross Product)

$$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\| \, \|\mathbf{b}\| \ \sin\gamma.$$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \mathbf{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}.$$

1.4 Scalar Triple Product

$$(\mathbf{a}\,\mathbf{b}\,\mathbf{c}) := \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

$$\big|(\mathbf{a}\,\mathbf{b}\,\mathbf{c})\big| = \text{volume of the parallelepiped with edges } \mathbf{a}, \mathbf{b}, \mathbf{c}.$$

$$\mathbf{a}, \mathbf{b}, \mathbf{c} \text{ are linearly independent } \iff (\mathbf{a}\,\mathbf{b}\,\mathbf{c}) \neq 0.$$

1.5 Derivatives

$$\mathbf{v}(\mathbf{t}) = [v_1(\mathbf{t}), v_2(\mathbf{t}), v_3(\mathbf{t})], \ \ \mathbf{t} = (t_1, \ldots, t_n).$$

$$\frac{\partial \mathbf{v}}{\partial t_m} = \frac{\partial v_1}{\partial t_m}\mathbf{i} + \frac{\partial v_2}{\partial t_m}\mathbf{j} + \frac{\partial v_3}{\partial t_m}\mathbf{k}.$$

1.6 Arc Length

$$\mathbf{r}(t) = [x(t), y(t), z(t)], \ \ s(t) = \int_a^t \|\mathbf{r}'(\tau)\| \, d\tau.$$

$$\text{Linear Element.} \quad ds^2 = d\mathbf{r} \cdot d\mathbf{r} = dx^2 + dy^2 + dz^2.$$

$$\text{Arc Length as Parameter.} \quad \mathbf{u}(s) = \mathbf{r}'(s).$$

1.7 Curvature

$$\kappa(s) = \left\| \frac{d\mathbf{u}}{ds} \right\| = \left\| \frac{d^2\mathbf{r}}{ds^2} \right\|.$$

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1.8 Torsion

$$\mathbf{p} = \frac{1}{\kappa}\frac{d\mathbf{u}}{ds}, \quad \mathbf{b} = \mathbf{u} \times \mathbf{p}.$$

$$|\tau| = \left\| \frac{d\mathbf{b}}{ds} \right\|, \quad \tau = -\,\mathbf{p} \cdot \frac{d\mathbf{b}}{ds}.$$

1.9 Gradient

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}, \ \ \nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}.$$

1.9.1 Directional Derivative.

$$D_{\mathbf{a}} f(P) \ = \ \frac{1}{\|\mathbf{a}\|}\,\mathbf{a} \cdot \nabla f(P) = \|\nabla f(P)\| \ \cos\gamma.$$

grad f points in the direction of maximum increase of f.

1.9.2 Gradient as Surface Normal.

$$f(x, y, z) = c, \ \ \mathbf{r}(t) = (x(t), y(t), z(t)), \ \ \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = 0.$$

$\mathbf{r}(t)$ is any differentiable curve on a level surface S through P, so $\nabla f(P)$ is orthogonal to every tangent vector at P and hence is a normal to S at P.

1.9.3 Laplacian.

$$\Delta \ = \ \nabla^2 \ = \ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

1.10 Divergence

$$\operatorname{div} \mathbf{v} = \nabla \cdot \mathbf{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}, \ \ \operatorname{div}(\nabla f) = \nabla^2 f.$$

Divergence is a scalar quantity independent of the particular Cartesian coordinates used.

1.11 Curl

$$\operatorname{curl} v = \nabla \times v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

Theorem 1.1 (Rotating Body and Curl). *For a rigid body rotating with constant angular velocity vector $\boldsymbol{\omega}$, the velocity field is $v = \boldsymbol{\omega} \times \mathbf{r}$ and $\operatorname{curl} v = 2\,\boldsymbol{\omega}$.*

Theorem 1.2 (Grad, Div, Curl Identities). *If f is a scalar field of class C^2 and v is a vector field of class C^2, then*

$$\operatorname{curl}(\nabla f) = \mathbf{0}, \quad \nabla \cdot (\operatorname{curl} v) = 0.$$

(Gradient fields are irrotational; the divergence of a curl is zero.)

Theorem 1.3 (Invariance of the Curl). *Length and direction of $\operatorname{curl} v$ are independent of the particular right-handed Cartesian coordinate system.*

2 Vector Integral Calculus

2.1 Line Integrals

$$\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = \int_C (F_1 dx + F_2 dy + F_3 dz).$$

Theorem 2.1 (Path independence $\iff$ potential). *Let* $\mathbf{F}$ *have continuous components on a domain* D*. The line integral is path independent in* D *if and only if there exists a scalar function* f *on* D *such that*

$$\mathbf{F} = \nabla f \quad \text{on } D,$$

in which case, for any $A, B \in D$ *and any curve* C *from* A *to* B*,*

$$\int_C \mathbf{F} \cdot d\mathbf{r} \;=\; f(B) - f(A).$$

Then, f *is called a potential for* $\mathbf{F}$*, and* $\mathbf{F}$ *is conservative.*

Theorem 2.2 (Path independence $\iff$ zero around closed curves). *The line integral is path independent in* D *if and only if*

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$$

for every closed curve C *contained in* D*.*

Theorem 2.3 (Path independence $\iff$ exactness). *The line integral is path independent in* D *if and only if the differential (Pfaffian) form* $F_1\, dx + F_2\, dy + F_3\, dz$ *has continuous coefficients and is exact in* D*, i.e. there exists* f *with*

$$F_1\, dx + F_2\, dy + F_3\, dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz = df \iff \mathbf{F} = \nabla f.$$

Theorem 2.4 (Criterion via curl and simple connectivity). *Let* F_1, F_2, F_3 *be continuous on* D *with continuous first partial derivatives. Then,*

(a) *If the Pfaffian form is exact in* D*, then*

$$\nabla \times \mathbf{F} = \mathbf{0}, \quad \text{i.e.} \quad \frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z},\ \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x},\ \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}.$$

(b) *If* $\nabla \times \mathbf{F} = \mathbf{0}$ *holds on* D *and* D *is simply connected, i.e. every closed curve in* D *can be continuously contracted to a point in* D*, then the Pfaffian form is exact in* D*.*

2.2 Green's Theorem in the Plane

Theorem 2.5 (Green's Theorem in the Plane). *Let* R *be a closed, bounded region in the* xy*-plane whose boundary* C *consists of finitely many smooth curves. Let* $F_1(x, y)$ *and* $F_2(x, y)$ *be functions that are continuous on a domain containing* R *and have continuous first partial derivatives there. Then,*

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx\, dy \;=\; \oint_C \big(F_1\, dx + F_2\, dy\big),$$

$$\iint_R (\nabla \times \mathbf{F}) \cdot \mathbf{k}\, dx\, dy \;=\; \oint_C \mathbf{F} \cdot d\mathbf{r},$$

where C *is oriented so that* R *remains on the left as one traverses* C*.*

proof. Assume R can be represented as

$$a \le x \le b, \quad u(x) \le y \le v(x),$$

with boundary pieces C^* (graph $y = u(x)$, oriented from $x = a$ to $x = b$) and C^{**} (graph $y = v(x)$, oriented from $x = b$ back to $x = a$).

$$\iint_R \frac{\partial F_1}{\partial y}\, dx\, dy = \int_a^b \left(\int_{u(x)}^{v(x)} \frac{\partial F_1}{\partial y}\, dy \right) dx = \int_a^b \big(F_1[v] - F_1[u]\big)\, dx$$

$$= -\int_{C^{**}} F_1\, dx - \int_{C^*} F_1\, dx = -\int_C F_1\, dx.$$

Similarly, representing R as

$$c \le y \le d, \quad p(y) \le x \le q(y),$$

$$\iint_R \frac{\partial F_2}{\partial x}\, dx\, dy = \int_c^d \big(F_2[q(y), y] - F_2[p(y), y]\big)\, dy = \int_C F_2\, dy.$$

So,

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx\, dy = \int_C F_1\, dx + \int_C F_2\, dy = \oint_C (F_1\, dx + F_2\, dy).$$

□

2.3 Surface Integrals

Surface $S \subset \mathbb{R}^3$ may be given as

$$z = f(x, y) \text{ or } g(x, y, z) = 0 \text{ or } \mathbf{r}(u, v) = [x(u, v),\ y(u, v),\ z(u, v)].$$

Treating u and v as coordinates on S, the parameter curves $u = \text{const}$ and $v = \text{const}$ meet at a point $P \in S$. Their tangent vectors are the partial derivatives $\frac{\partial \mathbf{r}}{\partial u}, \frac{\partial \mathbf{r}}{\partial v}$. Assume $\mathbf{r}_u$ and $\mathbf{r}_v$ are continuous and linearly independent. Then they span the tangent plane at P, and a unit normal vector is given by the cross product is

$$\mathbf{n} = \frac{\mathbf{N}}{\|\mathbf{N}\|} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{\|\mathbf{r}_u \times \mathbf{r}_v\|}.$$

If S is given implicitly by $g(x, y, z) = 0$ with differentiable g and $\nabla g \ne \mathbf{0}$ on S, then a unit normal at $(x, y, z) \in S$ is

$$\mathbf{n}(x, y, z) = \frac{\nabla g(x, y, z)}{\|\nabla g(x, y, z)\|}.$$

A surface is called *smooth* if its surface normal depends continuously on the point. It is *piecewise smooth* if it is a finite union of smooth pieces.

The oriented surface/flux integral of a vector field $\mathbf{F}$ over S is

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dA = \iint_R \mathbf{F}\big(\mathbf{r}(u, v)\big) \cdot \mathbf{N}(u, v)\, du\, dv,$$

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dA = \iint_S \big(F_1\, dy\, dz + F_2\, dz\, dx + F_3\, dx\, dy\big),$$

and the area element satisfies

$$dA = \|\mathbf{N}\|\, du\, dv = \|\mathbf{r}_u \times \mathbf{r}_v\|\, du\, dv.$$

For a graph $S:\ z = h(x, y)$ parameterized by $\mathbf{r}(x, y) = (x, y, h(x, y))$,

$$\iint_S F_3 \cos\gamma\, dA = \iint_{R_{xy}} F_3\big(x, y, h(x, y)\big)\, dx\, dy, \quad \text{when } \cos\gamma \ge 0,$$

$$\iint_S F_3 \cos\gamma\, dA = -\iint_{R_{xy}} F_3\big(x, y, h(x, y)\big)\, dx\, dy, \quad \text{when } \cos\gamma \le 0.$$

A smooth surface is *orientable* if one can choose a continuous unit normal field $\mathbf{n}$ globally on S. For a piecewise smooth surface $S = \bigcup S_k$ with smooth pieces S_k, S is orientable if the piecewise choices of normals agree across common boundaries so that the induced boundary orientations on shared edges are opposite and hence compatible. Some surfaces (e.g., the Möbius strip) are not orientable.

For a scalar field G, the unoriented surface integral is

$$\iint_S G(\mathbf{r})\,dA = \iint_R G\big(\mathbf{r}(u,v)\big)\,\|\mathbf{N}(u,v)\|\,du\,dv.$$

If R is simply connected and G is continuous on a domain containing R, then there exists $(u_0, v_0) \in R$ such that the mean-value relation holds

$$\iint_S G(\mathbf{r})\,dA = G\big(\mathbf{r}(u_0,v_0)\big)\,A(S),$$

where

$$A(S) \;=\; \iint_S dA \;=\; \iint_R \|\mathbf{r}_u \times \mathbf{r}_v\|\,du\,dv.$$

2.4 Divergence Theorem of Gauss

Let T be a closed, bounded region with piecewise smooth, orientable boundary surface $S = \partial T$ and outward unit normal n. If F is continuously differentiable on a domain containing T, then

$$\iiint_T \operatorname{div} F\,dV \;=\; \iint_S F \cdot n\,dA.$$

$$\iiint_T \left(\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}\right) dx\,dy\,dz = \iint_S (F_1\,dy\,dz+F_2\,dz\,dx+F_3\,dx\,dy).$$

Proof.

$$\iiint_T \frac{\partial F_3}{\partial z}\,dx\,dy\,dz = \iint_R \Big(F_3[x,y,h(x,y)] - F_3[x,y,g(x,y)]\Big)\,dx\,dy.$$

Triple integrals satisfy a mean value theorem: for continuous f on a bounded, simply connected T, there exists $Q = (x_0, y_0, z_0) \in T$ with

$$\iiint_T f\,dV = f(x_0,y_0,z_0)\,V(T).$$

$$\frac{1}{V(T)} \iint_{\partial T} F \cdot n\,dA \;=\; \operatorname{div} F(x_0,y_0,z_0).$$

$$\operatorname{div} F(P) = \lim_{d(T)\to 0} \frac{1}{V(T)} \iint_{\partial T} F \cdot n\,dA.$$

Laplace's equation

$$\Delta f \;=\; f_{xx} + f_{yy} + f_{zz} \;=\; 0$$

defines *harmonic* functions.

THEOREM 2.6 (BASIC PROPERTY OF HARMONIC FUNCTIONS). *If f is harmonic in a domain D and $S \subset D$ is any piecewise smooth closed orientable surface bounding a region $T \subset D$, then*

$$\iint_S \frac{\partial f}{\partial n}\,dA = \iint_S \nabla f \cdot n\,dA = \iiint_T \Delta f\,dV = 0.$$

THEOREM 2.7 (ZERO BOUNDARY ⇒ ZERO INTERIOR). *If f is harmonic in a domain D and $f = 0$ on a piecewise smooth closed orientable surface $S \subset D$ that bounds $T \subset D$, then $f \equiv 0$ in T.*

THEOREM 2.8 (UNIQUENESS FOR LAPLACE'S EQUATION / DIRICHLET PROBLEM)). *The* Dirichlet problem *seeks a function u in a region T solving $\Delta u = 0$ and taking prescribed values on $S = \partial T$. If a harmonic f in a domain containing $T \cup S$ is specified on S, then f is uniquely determined in T by its boundary values on S. Equivalently, if two harmonic functions agree on S, they agree throughout T.*

2.5 Stokes's Theorem

Let S be a piecewise smooth oriented surface with boundary $C = \partial S$ a piecewise smooth simple closed curve. If F has continuous first partials on a domain containing S, then

$$\iint_S (\operatorname{curl} F) \cdot n\,dA \;=\; \oint_C F \cdot \mathbf{t}\,ds \;=\; \oint_C (F_1\,dx + F_2\,dy + F_3\,dz).$$

If F is continuously differentiable on a simply connected domain D and $\operatorname{curl} F = 0$ in D, then for any closed curve $C \subset D$,

$$\oint_C F \cdot \mathbf{t}\,ds \;=\; \iint_S (\operatorname{curl} F) \cdot n\,dA \;=\; 0,$$

so line integrals $\int_A^B F \cdot dr$ are path independent in D. Conversely, path independence implies $\operatorname{curl} F = 0$ when F has continuous first partials.

3 Complex Differentiation

3.1 Complex Numbers and Their Geometric Representation

A complex number is an ordered pair (x, y) of real numbers,

$$z = (x, y),$$

where x is the *real part* and y the *imaginary part*, denoted $x = \Re z$ and $y = \Im z$. Two complex numbers are equal iff their real parts and imaginary parts are equal. The *imaginary unit* is $(0, 1)$ and is denoted by i.

$$z = x + iy \quad \text{for} \quad z = (x, y).$$

If $\Re z = 0$, then $z = iy$ is *purely imaginary.*

$$\frac{x_1 + iy_1}{x_2 + iy_2} = \frac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)} = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} \;+\; i\,\frac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2}.$$

The *complex conjugate* of $z = x + iy$ is $\overline{z} = x - iy$.

$$\overline{z + w} = \overline{z} + \overline{w}, \quad \overline{zw} = \overline{z}\,\overline{w}, \quad \overline{\left(\frac{z}{w}\right)} = \frac{\overline{z}}{\overline{w}} \;(w \neq 0), \quad \overline{\overline{z}} = z.$$

$$\Re z = \frac{z + \overline{z}}{2}, \quad \Im z = \frac{z - \overline{z}}{2i}.$$

3.2 Polar Form of Complex Numbers

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = r\big(\cos\theta + i\sin\theta\big).$$

$$|z| = r = \sqrt{x^2 + y^2}, \quad \tan\theta = \frac{y}{x}, \quad \theta = \arg z.$$

The *principal value* $\operatorname{Arg} z$ is defined by $-\pi < \operatorname{Arg} z \leq \pi$.

$$|z_1z_2| = |z_1|\,|z_2|, \quad \arg(z_1z_2) = \arg z_1 + \arg z_2, \quad \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}.$$

De Moivre's formula. For any integer $n \geq 0$,

$$\big(\cos\theta + i\sin\theta\big)^n = \cos(n\theta) + i\sin(n\theta).$$

Roots.

$$w_k = r^{1/n}\left(\cos\frac{\theta + 2\pi k}{n} + i\sin\frac{\theta + 2\pi k}{n}\right), \quad k = 0, 1, \ldots, n-1.$$

3.3 Derivative

A function f is *analytic/holomorphic in a domain D* if f is defined and differentiable at every point of D. We say that f is *analytic at a point $z_0 \in D$* if f is analytic in some neighborhood of z_0.

3.4 Cauchy–Riemann Equations. Laplace's Equation

$$f(z) = f(x+iy) = u(x,y) + i\,v(x,y).$$

Cauchy–Riemann (CR) equations provide a criterion for analyticity.

$$u_x = v_y, \quad u_y = -\,v_x,$$

whenever the first partial derivatives exist and are continuous on D. Writing $z = re^{i\theta}$ and $u = u(r,\theta)$, $v = v(r,\theta)$, the CR equations are

$$u_r \;=\; \frac{1}{r}\,v_\theta, \quad v_r \;=\; -\,\frac{1}{r}\,u_\theta \quad (r > 0).$$

Theorem 3.1 (Cauchy–Riemann Equations. Necessity). *Suppose f is differentiable at a point $z_0 = x + iy$. Then the first-order partial derivatives u_x, u_y, v_x, v_y exist at (x,y) and satisfy the Cauchy–Riemann equations at (x,y).*

Proof. By differentiability at z,

$$f'(z) = \lim_{\Delta z\to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\Delta z\to 0} \frac{\big[u(x+\Delta x, y+\Delta y) - u(x,y)\big] + i\big[v(x+\Delta x, y+\Delta y) - v(x,y)\big]}{\Delta x + i\,\Delta y}.$$

Set $\Delta y = 0$ and let $\Delta x \to 0$,

$$f'(z) \;=\; u_x(x,y) \;+\; i\,v_x(x,y).$$

Set $\Delta x = 0$ and let $\Delta y \to 0$,

$$f'(z) \;=\; -\,i\,u_y(x,y) \;+\; v_y(x,y).$$

□

Theorem 3.2 (Cauchy–Riemann Equations. Sufficiency). *If u, v are real-valued functions on a domain D with continuous first partial derivatives that satisfy the Cauchy–Riemann equations on D, then $f = u + iv$ is analytic on D.*

Theorem 3.3 (Real and imaginary parts are harmonic). *If $f = u + iv$ is analytic on a domain D, then both u and v satisfy Laplace's equation on D.*

$$u_{xx} + u_{yy} = 0, \quad v_{xx} + v_{yy} = 0.$$

In particular, u and v have continuous second partial derivatives on D.

Proof. Differentiate the CR equations

$$\partial_x(u_x = v_y) \;\Rightarrow\; u_{xx} = v_{yx}, \quad \partial_y(u_y = -v_x) \;\Rightarrow\; u_{yy} = -v_{xy}.$$

By equality of mixed partials $v_{xy} = v_{yx}$, we get $u_{xx} + u_{yy} = 0$. □

Harmonic conjugates. If u and v are harmonic on D and satisfy the CR equations on D, then $u + iv$ is analytic on D. In this case, v is called a *harmonic conjugate* of u on D, unique up to an additive real constant.

3.5 Exponential Function. Trigonometric and Hyperbolic Functions. Euler's Formula

For $z = x + iy$ we *define*

$$e^z \;=\; e^x\big(\cos y + i\sin y\big).$$

A function analytic for all z is called *entire*. Thus e^z is entire.

$$\cos z = \tfrac{1}{2}\big(e^{iz} + e^{-iz}\big), \quad \sin z = \frac{1}{2i}\big(e^{iz} - e^{-iz}\big).$$

Euler's formula remains valid for complex z.

$$e^{iz} = \cos z + i\sin z.$$

$$\cosh z = \tfrac{1}{2}(e^z + e^{-z}), \quad \sinh z = \tfrac{1}{2}(e^z - e^{-z}).$$

$$\cosh(iz) = \cos z, \quad \sinh(iz) = i\,\sin z.$$

$$\cos(iz) = \cosh z, \quad \sin(iz) = i\,\sinh z.$$

3.6 Logarithm. General Power. Principal Value

Since the argument of z is only determined up to multiples of 2π, the complex logarithm is *infinitely many-valued.*

$$\ln z = \ln r + i\,u \quad (r = |z|,\ u = \arg z).$$

The value corresponding to the principal argument $\operatorname{Arg} z$ is the *principal value*

$$\operatorname{Ln} z = \ln|z| + i\operatorname{Arg} z \quad (z \neq 0),$$

which is single-valued. All other values are

$$\ln z = \operatorname{Ln} z + 2n\pi i, \quad n \in \mathbb{Z}.$$

$$\ln(e^z) = z + 2n\pi i, \quad n \in \mathbb{Z}.$$

For each fixed $n \in \mathbb{Z}$, the branch

$$\ln_n z \;=\; \operatorname{Ln} z + 2n\pi i$$

defines an analytic function on $\mathbb{C} \setminus \{x \le 0\}$, with

$$(\ln_n z)' = \frac{1}{z} \quad (z \notin (-\infty, 0]).$$

Each such single-valued analytic determination is called a *branch* of $\ln z$. The negative real axis is a *branch cut*, and $n = 0$ gives the *principal branch.*
For complex c and $z \neq 0$,

$$z^c = e^{c\ln z}.$$

Because $\ln z$ is multivalued, z^c is in general multivalued. Its *principal value* is

$$z^c\big|_{\text{principal}} = e^{c\operatorname{Ln} z}.$$

If $c = n \in \mathbb{Z}_{\ge 1}$ or $c \in -\mathbb{Z}_{\ge 1}$, then z^c is single-valued. If $c = \frac{1}{n}$, then z^c assumes the n distinct nth roots. If $c = p/q$ is rational, one likewise gets finitely many values. If c is irrational real or complex, there are infinitely many values.

4 Complex Integration

4.1 Line Integral in the Complex Plane

Complex definite integrals are called complex *line integrals*

$$\int_C f(z)\,dz.$$

Here the integrand is integrated over a given curve C. The curve C in the complex plane is called the *path of integration.* We represent C by a parametric form

$$z(t) = x(t) + iy(t), \quad a \le t \le b.$$

We assume C to be *smooth,*

$$\dot{z}(t) = \frac{dz}{dt} = \dot{x}(t) + i\dot{y}(t).$$

Theorem 4.1 (Indefinite integration of analytic functions). *If f is analytic in a simply connected domain D, i.e., every simple closed curve in D encloses only points of D, then there exists an analytic function F in D with $F'(z) = f(z)$. For all paths in D joining z_0 to z_1,*

$$\int_{z_0}^{z_1} f(z)\,dz = F(z_1) - F(z_0).$$

Theorem 4.2 (Integration by use of the path). *Let C be a piecewise smooth path represented by $z = z(t)$, $a \le t \le b$, and let f be continuous on C. Then,*

$$\int_C f(z)\,dz \;=\; \int_a^b f\big(z(t)\big)\,\dot{z}(t)\,dt, \quad \dot{z}(t) = \frac{dz}{dt} = \dot{x}(t) + i\dot{y}(t).$$

In general, for a given f and endpoints z_0, z_1, the value of $\int_{z_0}^{z_1} f(z)\,dz$ depends on the chosen path between them.

Let L be the length of C and suppose $|f(z)| \le M$ on C. Then

$$\left|\int_C f(z)\,dz\right| \;\le\; ML.$$

Proof.

$$\big|S_n\big| = \left|\sum f(\zeta_m)\,\Delta z_m\right| \le \sum |f(\zeta_m)|\,|\Delta z_m| \le M \sum |\Delta z_m|.$$

As the mesh goes to zero, $\sum |\Delta z_m|$ tends to the arc length L of C. □

4.2 Cauchy's Integral Theorem

A bounded domain is called p-fold connected if its boundary consists of p disjoint closed sets. D has $(p-1)$ holes.

Theorem 4.3 (Cauchy's Integral Theorem). *f is analytic in a simply connected domain D, then for every simple closed path C in D,*

$$\int_C f(z)\,dz \;=\; 0.$$

Proof. Write $f = u + iv$ with real-valued u, v. Then,

$$\int_C f(z)\,dz = \int_C (u\,dx - v\,dy) \;+\; i\int_C (u\,dy + v\,dx).$$

If f is analytic, then $u_x = v_y$ and $u_y = -v_x$. By Green's theorem,

$$\int_C (u\,dx - v\,dy) = \iint_R \left(\tfrac{\partial(-v)}{\partial x} - \tfrac{\partial u}{\partial y}\right) dx\,dy = \iint_R (-v_x - u_y)\,dx\,dy = 0,$$

$$\int_C (u\,dy + v\,dx) \;=\; \iint_R \big(u_x - v_y\big)\,dx\,dy = 0.$$

□

Theorem 4.4 (Existence of an Indefinite Integral). *If f is analytic in a simply connected domain D, then there exists an analytic function F on D with*

$$F'(z) = f(z),$$

and for all $z_0, z_1 \in D$ and any path C in D from z_0 to z_1,

$$\int_C f(z)\,dz \;=\; F(z_1) - F(z_0).$$

Proof.

$$F(z) \;=\; \int_{z_0}^{z} f(\zeta)\,d\zeta,$$

$$\frac{F(z+\Delta z) - F(z)}{\Delta z} = \frac{1}{\Delta z}\int_z^{z+\Delta z} f(\zeta)\,d\zeta = f(z) + \frac{1}{\Delta z}\int_z^{z+\Delta z} \big(f(\zeta) - f(z)\big)\,d\zeta.$$

Since f is continuous, given $\varepsilon > 0$ there exists $\delta > 0$ with $|f(\zeta) - f(z)| < \varepsilon$ whenever $|\zeta - z| < \delta$. For $|\Delta z| < \delta$, ML–inequality gives

$$\left|\frac{1}{\Delta z}\int_z^{z+\Delta z} \big(f(\zeta) - f(z)\big)\,d\zeta\right| \le \frac{1}{|\Delta z|}\,\varepsilon\,|\Delta z| = \varepsilon,$$

□

Let D be doubly connected with outer boundary C_1 and inner boundary C_2, both positively oriented with respect to D. Suppose f is analytic in a domain containing D and its boundary curves. Then

$$\int_{C_1} f(z)\,dz \;=\; \int_{C_2} f(z)\,dz,$$

independently of whether the interior of C_2 belongs to D.

4.3 Cauchy's Integral Formula

Theorem 4.5 (Cauchy's Integral Formula). *Let f be analytic in a simply connected domain D. Then for any point z_0 in D and any simple closed path C in D that encloses z_0,*

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z - z_0}\,dz$$

the integration being taken counterclockwise.

Proof.

$$\int_C \frac{f(z)}{z - z_0}\,dz = f(z_0)\int_C \frac{dz}{z - z_0} + \int_C \frac{f(z) - f(z_0)}{z - z_0}\,dz.$$

The first term equals $2\pi i\, f(z_0)$. The integrand of the second integral is analytic, except at z_0. We can replace C by a small circle K of radius ρ and center z_0, without altering the value of the integral. Since f is analytic, it is continuous. Hence, an $\varepsilon > 0$ being given, we can find a $\delta > 0$ such that for all z in the disk $|z - z_0| < \delta$,

$$|f(z) - f(z_0)| < \varepsilon.$$

Choosing the radius ρ of K smaller than δ,

$$\left|\frac{f(z) - f(z_0)}{z - z_0}\right| \le \frac{\varepsilon}{\rho}$$

at each point of K. Hence, by the ML-inequality,

$$\left|\int_K \frac{f(z) - f(z_0)}{z - z_0}\,dz\right| \le \frac{\varepsilon}{\rho}\,2\pi\rho = 2\pi\varepsilon.$$

□

4.4 Derivatives of Analytic Functions

Theorem 4.6 (Derivatives of an Analytic Function). *If f is analytic in a domain D, then it has derivatives of all orders in D, which are then also analytic functions in D.*

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz, \quad n = 1, 2, \ldots,$$

where C is any simple closed path in D that encloses z_0 and whose full interior is in D, and we integrate counterclockwise around C.

Proof.

$$\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z} = \frac{1}{2\pi i\Delta z}\left(\int_C \frac{f(z)}{z-(z_0+\Delta z)}\,dz - \int_C \frac{f(z)}{z-z_0}\,dz\right)$$
$$= \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0-\Delta z)(z-z_0)}\,dz.$$

$$\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z} - \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^2}\,dz$$
$$= \frac{1}{2\pi i}\int_C \frac{f(z)\,\Delta z}{(z-z_0-\Delta z)(z-z_0)^2}\,dz.$$

Being analytic, the function f is continuous on C, hence bounded in absolute value, say,

$$|f(z)| \le K \quad \text{for all } z \in C.$$

Let d be the smallest distance from z_0 to the points of C. Then,

$$|z-z_0| \ge d \quad \Rightarrow \quad \frac{1}{|z-z_0|^2} \le \frac{1}{d^2}.$$
$$|z-(z_0+\Delta z)| \ge |z-z_0| - |\Delta z| \ge d - |\Delta z|.$$

If $|\Delta z| \le d/2$, then

$$|z-(z_0+\Delta z)| \ge d - \frac{d}{2} = \frac{d}{2} \quad \Rightarrow \quad \frac{1}{|z-(z_0+\Delta z)|} \le \frac{2}{d}.$$

Hence, on C,

$$\left|\frac{f(z)\,\Delta z}{(z-z_0-\Delta z)(z-z_0)^2}\right| \le \frac{K\,|\Delta z|}{(d/2)\,d^2} = \frac{2K}{d^3}|\Delta z|.$$

Let L be the length of C. If $|\Delta z| \le d/2$, then by the ML-inequality

$$\left|\int_C \frac{f(z)\,\Delta z}{(z-z_0-\Delta z)(z-z_0)^2}\,dz\right| \le \frac{2K}{d^3}|\Delta z|\,L.$$

□

Cauchy's Inequality.

$$\left|f^{(n)}(z_0)\right| \le \frac{n!}{2\pi}\cdot M \cdot \max_{z\in C}\frac{1}{|z-z_0|^{n+1}}\cdot(2\pi r) = \frac{n!M}{r^n}.$$

Theorem 4.7 (Liouville's Theorem). *If an entire function f is bounded in absolute value in the whole complex plane, then this function must be a constant.*

Proof. By assumption, $f(z)$ is bounded, say,

$$|f(z)| \le K \quad \text{for all } z.$$
$$|f'(z_0)| \le \frac{K}{r}.$$

Since f is entire, this holds for every $r > 0$, s.t. we can take r as large as we please and conclude that $f'(z_0) = 0$. Since z_0 is arbitrary,

$$f'(z) = 0 \quad \text{for all } z.$$

Writing $f = u + iv$, we obtain $u_x = v_x = 0$ for all z. By the Cauchy–Riemann equations, it then follows that $u_y = v_y = 0$ as well. Thus u and v are constant, and hence f is constant. □

Theorem 4.8 (Morera's Theorem). If f is continuous in a simply connected domain D and if

$$\int_C f(z)\,dz = 0$$

for every closed path C in D, then f is analytic in D.

Proof. As the integral of f around every closed path in D is zero,

$$F(z) = \int_{z_0}^{z} f(z^*)\,dz^*$$

is well-defined (path-independence) and as f is continuous,

$$F'(z) = f(z).$$

So, F is analytic. Then, the derivative of F is analytic. □

5 Power Series, Taylor Series

5.1 Sequences, Series, Convergence Tests

Theorem 5.1 (Sequences of the Real and the Imaginary Parts). *A sequence of complex numbers $z_1, z_2, \ldots, z_n, \ldots$ (where $z_n = x_n + iy_n$) converges to $c = a + ib$ if and only if the sequence of the real parts $\{x_n\}$ converges to a and the sequence of the imaginary parts $\{y_n\}$ converges to b.*

Theorem 5.2 (Real and Imaginary Parts). *A series $\sum_{m=1}^{\infty} z_m$ with $z_m = x_m + iy_m$ converges and has the sum $s = u + iv$ if and only if $\sum_{m=1}^{\infty} x_m$ converges and has the sum u and $\sum_{m=1}^{\infty} y_m$ converges and has the sum v.*

Theorem 5.3 (Divergence). *If a series $\sum_{m=1}^{\infty} z_m$ converges, then $z_m \longrightarrow 0$. Hence if this does not hold, the series diverges.*

Theorem 5.4 (Cauchy's Convergence Principle for Series). *A series $\sum_{m=1}^{\infty} z_m$ is convergent if and only if for every given $\varepsilon > 0$ we can find an N (which depends on ε in general) such that*

$$|z_{n+1} + z_{n+2} + \cdots + z_{n+p}| < \varepsilon \quad \textit{for every } n \ge N \textit{ and } p = 1, 2, \ldots.$$

Theorem 5.5 (Comparison Test). *If a series $\sum_{m=1}^{\infty} z_m$ is given and we can find a convergent series $\sum_{m=1}^{\infty} b_m$ with nonnegative real terms such that*

$$|z_m| \le b_m \quad \textit{for all } m,$$

then the given series $\sum_{m=1}^{\infty} z_m$ converges, even absolutely.

Theorem 5.6 (Geometric Series). *The geometric series*

$$\sum_{m=0}^{\infty} q^m = 1 + q + q^2 + \ldots$$

converges with the sum

$$\frac{1}{1-q}$$

if $|q| < 1$ and diverges if $|q| \ge 1$.

Theorem 5.7 (Ratio Test, Form I). *If a series $\sum_{n=1}^{\infty} z_n$ with $z_n \neq 0$ has the property that for every n greater than some N,*

$$\frac{|z_{n+1}|}{|z_n|} \leq q < 1,$$

where q is fixed and independent of n, then this series converges absolutely. If for every n greater than some N,

$$\frac{|z_{n+1}|}{|z_n|} \geq 1,$$

the series diverges.

Theorem 5.8 (Ratio Test, Form II). *If a series $\sum_{n=1}^{\infty} z_n$ with $z_n \neq 0$ is such that*

$$\lim_{n\to\infty} \left|\frac{z_{n+1}}{z_n}\right| = L,$$

then

(a) If $L < 1$, the series converges absolutely.
(b) If $L > 1$, the series diverges.
(c) If $L = 1$, the series may converge or diverge.

Theorem 5.9 (Root Test, Form I). *If a series $\sum_{n=1}^{\infty} z_n$ is such that for every n greater than some N,*

$$\sqrt[n]{|z_n|} \leq q < 1,$$

where q is fixed and independent of n, this series converges absolutely. If for infinitely many n,

$$\sqrt[n]{|z_n|} \geq 1,$$

the series diverges.

Theorem 5.10 (Root Test, Form II). *If a series $\sum_{n=1}^{\infty} z_n$ is such that*

$$\lim_{n\to\infty} \sqrt[n]{|z_n|} = L,$$

then

(a) The series converges absolutely if $L < 1$.
(b) The series diverges if $L > 1$.
(c) If $L = 1$, the test fails.

5.2 Power Series

A **power series** in powers of $(z - z_0)$ is a series of the form

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots,$$

where z is a complex variable, the a_n are complex or real constants called the *coefficients*, and z_0 is a complex or real)constant called the *center* of the series.

Theorem 5.11 (Convergence of a Power Series). *Consider the power series*

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

(a) *The series converges at its center z_0.*
(b) *If the series converges at some point $z_1 \neq z_0$, then it converges absolutely for every z that is closer to z_0 than z_1 is, that is,*

$$|z - z_0| < |z_1 - z_0|.$$

(c) *If the series diverges at some point z_2, then it diverges for every z that is farther from z_0 than z_2 is, that is,*

$$|z - z_0| > |z_2 - z_0|.$$

Proof. (b)

$$a_n (z_1 - z_0)^n \to 0 \quad (n \to \infty),$$

and the sequence of partial sums is bounded. There exists $M > 0$ s.t.

$$\left|a_n (z_1 - z_0)^n\right| \leq M \quad \text{for all } n = 0, 1, 2, \ldots.$$

Now take any z satisfying $|z - z_0| < |z_1 - z_0|$. Then

$$\left|a_n (z - z_0)^n\right| = \left|a_n (z_1 - z_0)^n\right| \left|\frac{z - z_0}{z_1 - z_0}\right|^n \leq M \left|\frac{z - z_0}{z_1 - z_0}\right|^n.$$

Summing over $n \geq 1$ we obtain

$$\sum_{n=1}^{\infty} \left|a_n (z - z_0)^n\right| \leq M \sum_{n=1}^{\infty} \left|\frac{z - z_0}{z_1 - z_0}\right|^n.$$

Our assumption $|z - z_0| < |z_1 - z_0|$ implies

$$\left|\frac{z - z_0}{z_1 - z_0}\right| < 1.$$

Hence the geometric series on the right-hand side converges. By the comparison test, the series on the left-hand side converges absolutely at z.

(c) Suppose the statement were false. Then there would exist a point z_3 with

$$|z_3 - z_0| > |z_2 - z_0|$$

such that the series converges at z_3. But then part (b) applied to the point z_3 would imply convergence at every point closer to z_0 than z_3, in particular at z_2. This contradicts the assumption that the series diverges at z_2. □

The circle

$$|z - z_0| = R$$

is called the **circle of convergence** of the power series, and R is called the **radius of convergence**. In general, nothing definite can be said about what happens on the circle itself, i.e. for $|z - z_0| = R$.

Theorem 5.12 (Radius of Convergence R). *Suppose that the sequence*

$$\left\{\left|\frac{a_{n+1}}{a_n}\right|\right\}_{n=1}^{\infty}$$

converges and has limit L^*. Then:

- *If $L^* = 0$, then $R = \infty$, i.e. the power series converges for all z.*
- *If $0 < L^* < \infty$, then*

  $$R = \frac{1}{L^*},$$

  the Cauchy–Hadamard formula.
- *If $L^* = \infty$, then $R = 0$, i.e. the series converges only at z_0.*

Proof.

$$\lim_{n\to\infty} \left|\frac{a_{n+1}(z - z_0)^{n+1}}{a_n (z - z_0)^n}\right| = L^* \, |z - z_0|.$$

If $0 < L^* < \infty$, the ratio test says that the series converges when $L^*|z - z_0| < 1$ and diverges when $L^*|z - z_0| > 1$. Thus the boundary between convergence and divergence is given by $L^*|z - z_0| = 1$, so the radius of convergence is

$$R = \frac{1}{L^*}.$$

□

5.3 Functions Given by Power Series

Theorem 5.13 (Continuity of the Sum of a Power Series). *If a function f can be represented by a power series with radius of convergence $R > 0$, then f is continuous at $z = 0$.*

Proof. Let $S \neq 0$ be its sum. Then for $|z| \leq d$ and $d \leq r$ we have

$$|f(z) - a_0| = \left|\sum_{n=1}^{\infty} a_n z^n\right| \leq \sum_{n=1}^{\infty} |a_n|\,|z|^n \leq |z| \sum_{n=1}^{\infty} |a_n| r^{n-1} = |z|\, S \leq dS.$$

If we choose $d \leq P/S$, for $S > 0$, then $dS \leq P$. Hence $|f(z) - a_0| \leq P$ whenever $|z| < d$. □

Theorem 5.14 (Identity Theorem for Power Series; Uniqueness). *Let the power series*

$$\sum_{n=0}^{\infty} a_n z^n \quad \text{and} \quad \sum_{n=0}^{\infty} b_n z^n$$

both be convergent for $|z| < R$, where R is positive, and let them both have the same sum $f(z)$ for all these z. Then,

$$a_n = b_n, \quad n = 0, 1, 2, \ldots.$$

Hence if a function can be represented by a power series with any center z_0, this representation is unique.

Proof. We proceed by induction. By assumption,

$$a_0 + a_1 z + a_2 z^2 + \cdots = b_0 + b_1 z + b_2 z^2 + \cdots \quad (|z| < R).$$

The sums of these two power series are continuous at $z = 0$. Hence, if we consider $z \longrightarrow 0$ on both sides,

$$a_0 = b_0.$$

Now assume that $a_n = b_n$ for $n = 0, 1, \ldots, m$. Then,

$$a_{m+1} + a_{m+2} z + a_{m+3} z^2 + \cdots = b_{m+1} + b_{m+2} z + b_{m+3} z^2 + \cdots.$$

By letting $z \longrightarrow 0$ we conclude from this that

$$a_{m+1} = b_{m+1}.$$

□

Theorem 5.15 (Termwise Differentiation of a Power Series). *The derived series of a power series has the same radius of convergence as the original series.*

Proof.

$$\lim_{n\to\infty} \left|\frac{n a_n}{(n+1) a_{n+1}}\right| = \lim_{n\to\infty} \frac{n}{n+1} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|.$$

□

Theorem 5.16 (Termwise Integration of Power Series). *The power series*

$$\sum_{n=0}^{\infty} \frac{a_n}{n+1} z^{n+1} = a_0 z + \frac{a_1}{2} z^2 + \frac{a_2}{3} z^3 + \cdots,$$

has the same radius of convergence R as the original series.

Theorem 5.17 (Analytic Functions; Their Derivatives). *A power series with a nonzero radius of convergence R represents an analytic function at every point interior to its circle of convergence. The derivatives of this function are obtained by differentiating the original series term by term. All the series thus obtained have the same radius of convergence as the original series. Hence, by the first statement, each of them represents an analytic function.*

Proof.

$$\frac{f(z + \Delta z) - f(z)}{\Delta z} - f'(z) = \sum_{n=2}^{\infty} a_n \left(\frac{(z + \Delta z)^n - z^n}{\Delta z} - n z^{n-1}\right).$$

$$= \Delta z \sum_{n=2}^{\infty} a_n \left[(z+\Delta z)^{n-2} + 2z(z+\Delta z)^{n-3} + \cdots + (n-2) z^{n-3}(z+\Delta z) + (n-1) z^{n-2}\right].$$

The brackets contain $n - 1$ terms, and the largest coefficient is $n - 1$. Since $n - 1 \leq n(n-1)$ for $n \geq 2$, we see that for $|z| \leq R_0$ and $|z + \Delta z| \leq R_0$, where $R_0 < R$, the absolute value cannot exceed

$$|\Delta z| \sum_{n=2}^{\infty} |a_n|\, n(n-1)\, R_0^{n-2}.$$

This series is the second derived series at $z = 0$ and converges absolutely. Let the sum be $K(R_0)$. Then,

$$\left|\frac{f(z + \Delta z) - f(z)}{\Delta z} - f'(z)\right| \leq |\Delta z|\, K(R_0).$$

□

5.4 Taylor and Maclaurin Series

The Taylor series of a function f is

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

where

$$a_n = \frac{1}{n!} f^{(n)}(z_0),$$

or,

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z^*)}{(z^* - z_0)^{n+1}}\, dz^*.$$

We integrate counterclockwise around a simple closed path C that contains z_0 in its interior and is such that f is analytic in a domain containing C. A *Maclaurin series* is a Taylor series with center $z_0 = 0$. The remainder of the Taylor series after the term $a_n(z - z_0)^n$ is

$$R_n(z) = \frac{(z - z_0)^{n+1}}{2\pi i} \int_C \frac{f(z^*)}{(z^* - z_0)^{n+1}(z^* - z)}\, dz^*.$$

Theorem 5.18 (Taylor's Theorem). *Let f be analytic in a domain D, and let z_0 be any point in D. Then there exists precisely one Taylor series with center z_0 that represents $f(z)$. This representation is valid* *in the largest open disk with center z_0 in which f is analytic. The coefficients a_n satisfy the inequality*

$$|a_n| \le \frac{M}{r^n},$$

where M is the maximum of $|f|$ on a circle $|z - z_0| = r$ in D whose interior is also in D.

Proof.

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(z^*)}{z^* - z} \, dz^*,$$

where z lies inside C, for which we take a circle of radius r with center z_0 and interior in D.

$$\frac{1}{z^* - z} = \frac{1}{z^* - z_0 - (z - z_0)} = \frac{1}{z^* - z_0} \, \frac{1}{1 - \dfrac{z - z_0}{z^* - z_0}}.$$

We use in the form

$$\frac{1}{1-q} = 1 + q + \cdots + q^n + \frac{q^{n+1}}{1-q}, \quad n \ge 0.$$

$$\frac{1}{z^* - z} = \frac{1}{z^* - z_0} \left[1 + \frac{z - z_0}{z^* - z_0} + \cdots + \left(\frac{z - z_0}{z^* - z_0} \right)^n + \frac{\left(\dfrac{z - z_0}{z^* - z_0} \right)^{n+1}}{1 - \dfrac{z - z_0}{z^* - z_0}} \right]$$

$$= \sum_{k=0}^{n} \frac{(z - z_0)^k}{(z^* - z_0)^{k+1}} + \frac{(z - z_0)^{n+1}}{(z^* - z_0)^{n+1}(z^* - z)}.$$

$$f(z) = \frac{1}{2\pi i} \int_C f(z^*) \left[\sum_{k=0}^{n} \frac{(z - z_0)^k}{(z^* - z_0)^{k+1}} + \frac{(z - z_0)^{n+1}}{(z^* - z_0)^{n+1}(z^* - z)} \right] dz^*$$

$$= \sum_{k=0}^{n} (z - z_0)^k \, \frac{1}{2\pi i} \int_C \frac{f(z^*)}{(z^* - z_0)^{k+1}} \, dz^* + R_n(z).$$

Since z^* lies on C, whereas z lies inside C, we have $|z^* - z_0| = r$ and $|z - z_0| < r$. Since f is analytic inside and on C, it is bounded there, and so is the function $f(z^*)/(z^* - z_0)^{n+1}(z^* - z)$.

$$\left| \frac{f(z^*)}{(z^* - z_0)^{n+1}(z^* - z)} \right| \le \frac{\tilde{M}}{r^{n+1}}$$

for all z^* on C. Also, C has radius r and length $2\pi r$.

$$|R_n(z)| \le \frac{|z - z_0|^{n+1}}{2\pi} \int_C \left| \frac{f(z^*)}{(z^* - z_0)^{n+1}(z^* - z)} \right| |dz^*|$$

$$\le \frac{|z - z_0|^{n+1}}{2\pi} \cdot \frac{\tilde{M}}{r^{n+1}} \cdot 2\pi r = M \left(\frac{|z - z_0|}{r} \right)^{n+1}.$$

Now $|z - z_0| < r$, so that $\dfrac{|z - z_0|}{r} < 1$. □

Theorem 5.19 (Relation to the Previous Section). *A power series with a nonzero radius of convergence is the Taylor series of its sum.*

Proof. Given the power series

$$f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + a_3(z - z_0)^3 + \cdots,$$

$$f^{(n)}(z_0) = n! \, a_n.$$

□

6 Laurent Series. Residue Integration

6.1 Laurent Series

Theorem 6.1 (Laurent's Theorem). *Let $f(z)$ be analytic in a domain containing two concentric circles C_1 and C_2 with center z_0 and the annulus between them. Then f can be represented by*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n},$$

the Laurent series consisting of nonnegative and negative powers. The coefficients of this Laurent series are given by the integrals

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z^*)}{(z^* - z_0)^{n+1}} \, dz^*, \quad b_n = \frac{1}{2\pi i} \int_C (z^* - z_0)^{n-1} f(z^*) \, dz^*,$$

taken counterclockwise around any simple closed path C that lies in the annulus and encircles the inner circle.

In the important special case that z_0 is the only singular point of f inside C_2, this circle can be shrunk to the point z_0, giving convergence in a disk $0 < |z - z_0| < R$. In this case the series or finite sum of the negative powers is called the principal part *of $f(z)$ at z_0 or of that Laurent series.*

We may write

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n,$$

denoting by a_n all coefficients with indices $n = 0, \pm 1, \pm 2, \ldots$, where

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z^*)}{(z^* - z_0)^{n+1}} \, dz^* \quad (n = 0, \pm 1, \pm 2, \ldots).$$

Proof. (a) *The nonnegative powers.*

$$f(z) = g(z) + h(z) = \frac{1}{2\pi i} \int_{C_1} \frac{f(z^*)}{z^* - z} \, dz^* - \frac{1}{2\pi i} \int_{C_2} \frac{f(z^*)}{z^* - z} \, dz^*.$$

Here z is any point in the given annulus and we integrate counterclockwise over both C_1 and C_2.

$$g(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad a_n = \frac{1}{2\pi i} \int_{C_1} \frac{f(z^*)}{(z^* - z_0)^{n+1}} \, dz^*.$$

Here we can replace C_1 by C by the principle of deformation of path, since the point $z^* = z_0$ where the integrand is not analytic, is not a point of the annulus.

(b) *The negative powers.* Because $|z - z_0| > |z^* - z_0|$ for $z^* \in C_2$,

$$\frac{1}{z^* - z} = -\frac{1}{z - z^*} = -\frac{1}{z - z_0} \, \frac{1}{1 - \dfrac{z^* - z_0}{z - z_0}} = -\frac{1}{z - z_0} \sum_{n=0}^{\infty} \left(\frac{z^* - z_0}{z - z_0} \right)^n.$$

$$h(z) = -\frac{1}{2\pi i} \int_{C_2} \frac{f(z^*)}{z^* - z} \, dz^*$$

$$= \frac{1}{2\pi i} \int_{C_2} \frac{f(z^*)}{z - z_0} \sum_{n=0}^{\infty} \left(\frac{z^* - z_0}{z - z_0} \right)^n dz^*$$

$$= \frac{1}{2\pi i} \sum_{n=0}^{\infty} \frac{1}{(z - z_0)^{n+1}} \int_{C_2} (z^* - z_0)^n f(z^*) \, dz^*.$$

$$h(z) = \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} + R_n^*(z),$$

where

$$b_n = \frac{1}{2\pi i}\int_{C_2}(z^* - z_0)^{n-1} f(z^*)\,dz^*,$$

and $R_n^*(z)$ denotes the remainder after n terms. We can integrate over C instead of C_2 by deformation of path.

(c) *Convergence.* The function $f(z^*)$ is bounded in absolute value, say, $|f(z^*)| \le M$ for all $z^* \in C_2$. $|z - z^*| \ge |z - z_0| - |z^* - z_0| > 0$.

$$|R_n^*(z)| \le \frac{\tilde{M}L}{2\pi}\left(\frac{r_2}{|z - z_0|}\right)^{n+1},$$

where $L = 2\pi r_2$ and $r_2 = |z^* - z_0|$. □

6.2 Singularities and Zeros. Infinity

We say that a function f is *singular* or has a *singularity* at a point z_0 if f is not analytic, perhaps not even defined at z_0, but every neighborhood of z_0 contains points at which f is analytic. In this case z_0 is called a *singular point* of f. A singular point z_0 of f is called an *isolated singularity* if f has a neighborhood of z_0 in which there are no other singularities of f. Isolated singularities at z_0 can be classified by means of the Laurent series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}, \quad 0 < |z - z_0| < R,$$

valid in a punctured neighborhood of the singular point. The first series is analytic at z_0. The second series, consisting of the negative powers, is called the *principal part* of the Laurent expansion. If the principal part has only finitely many terms, it is of the form

$$\frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \cdots + \frac{b_m}{(z - z_0)^m}, \quad b_m \ne 0.$$

Then the singularity of f at z_0 is called a *pole* of order m. A pole of order 1 is also called a *simple pole.* If the principal part has infinitely many terms, we say that f has at z_0 an *isolated essential singularity.*

Theorem 6.2 (Poles). *Let f be analytic and have a pole at z_0. Then,*

$$|f(z)| \to \infty \quad \text{as } z \to z_0$$

along any path.

Theorem 6.3 (Picard's Theorem). *Let f be analytic and have an isolated essential singularity at a point z_0. Then, in every neighborhood of z_0, the function f assumes every complex value, with at most one exceptional value.*

We say that f has a *removable singularity* at z_0 if f is not analytic at z_0, but can be made analytic by assigning a suitable value to $f(z_0)$.

Let f be analytic in a domain D. $z_0 \in D$ is called a *zero* of f if

$$f(z_0) = 0.$$

The zero at z_0 is said to have *order n* if

$$f(z_0) = f'(z_0) = \cdots = f^{(n-1)}(z_0) = 0, \quad f^{(n)}(z_0) \ne 0.$$

A zero of order 1 is also called a *simple zero.* Suppose z_0 is an nth-order zero of f. Then, in the Taylor series of f about z_0,

$$f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots,$$

we have

$$a_0 = a_1 = \cdots = a_{n-1} = 0, \quad a_n \ne 0,$$

so that

$$\begin{aligned} f(z) &= a_n(z - z_0)^n + a_{n+1}(z - z_0)^{n+1} + a_{n+2}(z - z_0)^{n+2} + \cdots \\ &= (z - z_0)^n\left[a_n + a_{n+1}(z - z_0) + a_{n+2}(z - z_0)^2 + \cdots\right]. \end{aligned}$$

Theorem 6.4 (Zeros). *Let f be analytic in a domain D. Then every zero of f is isolated, that is, each zero has a neighborhood in which it is the only zero of f.*

Proof. Let z_0 be an nth-order zero of f. Then,

$$f(z) = (z - z_0)^n g(z),$$

where $g(z_0) = a_n \ne 0$. Since g is continuous and nonzero at z_0, there is a neighborhood of z_0 in which $g(z) \ne 0$. □

Theorem 6.5 (Poles and Zeros). *Let f be analytic at z_0 and have a zero of order n at z_0. Then, $1/f$ has a pole of order n at z_0. Moreover, if h is analytic at z_0 and $h(z_0) \ne 0$, then $h(z)/f(z)$ also has a pole of order n at z_0.*

Consider a sphere S of diameter 1 that touches the complex z-plane at the origin, the South Pole. Let N be the North Pole, diametrically opposite the origin. For each point P in the complex plane, representing a complex number z, draw the line segment PN. It intersects the sphere S at a point P^*. This gives a one-to-one correspondence between points z in the plane and points P^* on the sphere, except for N, which does not correspond to any finite point in the plane. We therefore introduce an additional point, denoted by ∞, and declare N to be its image. The complex plane plus this point ∞ is called the *extended complex plane.* The mapping described above is the *stereographic projection* of the extended complex plane onto the *Riemann sphere.* Consider the function

$$g(w) = f\left(\frac{1}{w}\right)$$

near $w = 0$. We say that f is *analytic at infinity* if g is analytic at $w = 0$, and that f is *singular at infinity* if g has a singularity at $w = 0$. We define the value of f at infinity by

$$f(\infty) = g(0) = \lim_{w \to 0} f\left(\frac{1}{w}\right) = \lim_{z \to \infty} f(z),$$

provided the limit exists.

An *entire* function is one that is analytic everywhere in the finite complex plane. Liouville's theorem implies that every bounded entire function must be constant. Hence, any nonconstant entire function is unbounded and therefore singular at infinity. In fact, a nonconstant entire function has either

- a pole at infinity, this happens precisely for polynomials, or
- an essential singularity at infinity, for nonpolynomial entire functions.

6.3 Residue Integration Method

$$b_1 = \frac{1}{2\pi i}\int_C f(z)\,dz.$$

The coefficient b_1 is called the *residue* of f at z_0 and we denote it by

$$b_1 = \operatorname{Res}_{z=z_0} f(z).$$

A first formula for the residue at a simple pole is

$$\operatorname{Res}_{z=z_0} f(z) = \lim_{z \to z_0} (z - z_0)\, f(z).$$

A second formula for the residue at a simple pole is

$$\operatorname{Res}_{z=z_0} f(z) = \operatorname{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}.$$

In this formula, we assume that $f(z) = p(z)/q(z)$ with $p(z_0) \neq 0$ and q has a simple zero at z_0.

Proof.

$$q(z) = (z - z_0) q'(z_0) + \frac{(z - z_0)^2}{2!}\, q''(z_0) + \cdots .$$

$$\begin{aligned}
\operatorname{Res}_{z=z_0} f(z) &= \lim_{z \to z_0} (z - z_0)\, \frac{p(z)}{q(z)} \\
&= \lim_{z \to z_0} \frac{(z - z_0) p(z)}{(z - z_0)\left[q'(z_0) + (z - z_0)\frac{q''(z_0)}{2!} + \cdots\right]} \\
&= \lim_{z \to z_0} \frac{p(z)}{q'(z_0) + (z - z_0)\frac{q''(z_0)}{2!} + \cdots} \\
&= \frac{p(z_0)}{q'(z_0)},
\end{aligned}$$

□

The residue of $f(z)$ at an mth-order pole at z_0 is given by

$$\operatorname{Res}_{z=z_0} f(z) = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} \left[(z - z_0)^m f(z) \right].$$

Proof.

$$f(z) = \frac{b_m}{(z - z_0)^m} + \frac{b_{m-1}}{(z - z_0)^{m-1}} + \cdots + \frac{b_1}{z - z_0} + a_0 + a_1 (z - z_0) + \cdots ,$$

where $b_m \neq 0$.

$$(z - z_0)^m f(z) = b_m + b_{m-1}(z - z_0) + \cdots + b_1 (z - z_0)^{m-1} + a_0 (z - z_0)^m + \cdots .$$

$$b_1 = \frac{g^{(m-1)}(z_0)}{(m-1)!} = \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left[(z - z_0)^m f(z) \right] \Big|_{z=z_0} .$$

□

Theorem 6.6 (Residue Theorem). *Let f be analytic inside a simple closed path C and on C, except for finitely many singular points $z_1, \ldots, z_k$ inside C. Then,*

$$\int_C f(z)\, dz = 2\pi i \sum_{j=1}^{k} Res_{z=z_j} f(z).$$

Proof. We enclose each of the singular points z_j in a circle C_j with radius small enough that the k circles and C are all disjoint and that all C_j lie inside C. Then $f(z)$ is analytic in the resulting multiply connected domain D bounded by C and the C_j, and on the entire boundary of D. By Cauchy's integral theorem, the integral of f around the boundary of D is zero.

$$\int_C f(z)\, dz - \int_{C_1} f(z)\, dz - \int_{C_2} f(z)\, dz - \cdots - \int_{C_k} f(z)\, dz = 0.$$

Hence,

$$\int_C f(z)\, dz = \int_{C_1} f(z)\, dz + \int_{C_2} f(z)\, dz + \cdots + \int_{C_k} f(z)\, dz.$$

$$\int_{C_j} f(z)\, dz = 2\pi i \operatorname{Res}_{z=z_j} f(z), \quad j = 1, \ldots, k.$$

$$\int_C f(z)\, dz = \sum_{j=1}^{k} 2\pi i \operatorname{Res}_{z=z_j} f(z) = 2\pi i \sum_{j=1}^{k} \operatorname{Res}_{z=z_j} f(z).$$

□

6.4 Residue Integration of Real Integrals

6.4.1 *Integrals of sin and cos.*

$$J = \int_0^{2\pi} F(\cos u, \sin u)\, du,$$

where F is a real rational function and is finite. Setting

$$z = e^{iu}, \quad 0 \le u \le 2\pi,$$

we obtain

$$\cos u = \frac{1}{2}\left(z + \frac{1}{z}\right), \quad \sin u = \frac{1}{2i}\left(z - \frac{1}{z}\right),$$

and

$$\frac{dz}{du} = i e^{iu} = iz, \quad du = \frac{dz}{iz}.$$

$$J = \int_0^{2\pi} F(\cos u, \sin u)\, du = \int_C f(z)\, \frac{dz}{iz},$$

where C is the unit circle $|z| = 1$. Thus, integrals can be transformed into contour integrals and evaluated by the residue theorem.

6.4.2 *Improper Integrals.*

$$\int_{-\infty}^{\infty} f(x)\, dx,$$

When it exists, it is called the *Cauchy principal value* of the integral.

$$\text{pr. v.} \int_{-\infty}^{\infty} f(x)\, dx.$$

It may exist even if the two limits do not exist separately. Consider the corresponding contour integral

$$\int_C f(z)\, dz,$$

around the path C consisting of the interval $[-R, R]$ on the real axis and the semicircle S of radius $R > 0$ in the upper half-plane, counterclockwise. Since f is rational, it has only finitely many poles in the upper half-plane, and by choosing R large enough, we can ensure that C encloses all those poles.

$$\int_C f(z)\, dz = \int_{-R}^{R} f(x)\, dx + \int_S f(z)\, dz = 2\pi i \sum \operatorname{Res} f(z).$$

We set

$$z = Re^{iu}, \quad 0 \le u \le \pi,$$

then $|z| = R$ and, by the degree assumption, for sufficiently large R there exist constants k, R_0 such that

$$|f(z)| \le \frac{k}{|z|^2} = \frac{k}{R^2}, \quad |z| = R \ge R_0.$$

$$\left| \int_S f(z)\, dz \right| \le \pi R \cdot \frac{k}{R^2} = \frac{k\pi}{R} \xrightarrow{R \to \infty} 0.$$

Thus,

$$\int_{-\infty}^{\infty} f(x)\, dx = 2\pi i \sum_{\Im z > 0} \operatorname{Res} f(z).$$

6.4.3 Fourier Integrals.

$$\int_{-\infty}^{\infty} f(x)\cos sx\,dx, \quad \int_{-\infty}^{\infty} f(x)\sin sx\,dx, \quad s \in \mathbb{R}.$$

We consider, for $s > 0$, the contour integral

$$\int_C f(z)e^{isz}\,dz$$

over the same contour C as above.

$$\int_{-\infty}^{\infty} f(x)e^{isx}\,dx = 2\pi i \sum_{\Im z>0} \operatorname{Res}\big(f(z)e^{isz}\big), \quad s > 0.$$

$$\int_{-\infty}^{\infty} f(x)\cos sx\,dx = 2\pi\,\Im \sum_{\Im z>0} \operatorname{Res}\big(f(z)e^{isz}\big),$$

$$\int_{-\infty}^{\infty} f(x)\sin sx\,dx = 2\pi\,\Re \sum_{\Im z>0} \operatorname{Res}\big(f(z)e^{isz}\big), \quad s > 0.$$

6.4.4 Simple Poles on the Real Axis. We now consider an improper integral

$$\int_A^B f(x)\,dx,$$

whose integrand becomes infinite at a point a in the interval of integration, $A < a < B$, that is,

$$\lim_{x\to a} |f(x)| = \infty.$$

Theorem 6.7 (Simple Poles on the Real Axis). *If f has a simple pole at $z = a$ on the real axis, then*

$$\lim_{r\to 0} \int_{C_2} f(z)\,dz = i\pi \operatorname{Res}_{z=a} f(z),$$

where C_2 is the semicircle of radius r centered at a in the upper half-plane and traversed counterclockwise from $a - r$ to $a + r$.

Proof.

$$f(z) = \frac{b_1}{z-a} + g(z), \quad b_1 = \operatorname{Res}_{z=a} f(z),$$

where g is analytic on and inside C_2. Hence,

$$\int_{C_2} f(z)\,dz = \int_{C_2} \frac{b_1}{z-a}\,dz + \int_{C_2} g(z)\,dz.$$

On C_2 we have $z = a + re^{iu}$, $0 \le u \le \pi$, so

$$\int_{C_2} \frac{b_1}{z-a}\,dz = b_1 \int_0^{\pi} \frac{1}{re^{iu}}\left(ire^{iu}\right)du = b_1 \int_0^{\pi} i\,du = i\pi b_1.$$

Since g is analytic on and inside C_2, it is bounded by M,

$$\left|\int_{C_2} g(z)\,dz\right| \le (\text{length of } C_2)\cdot \max_{C_2} |g(z)| \le \pi r M \xrightarrow{r\to 0} 0.$$

□

For sufficiently large R we consider a contour consisting of

- the segment $[-R, R]$ of the real axis,
- the semicircle S of radius R in the upper half-plane, and
- small semicircles above each simple pole of f on the real axis.

Therefore,

$$\text{pr. v.} \int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum_{\Im z>0} \operatorname{Res} f(z) + \pi i \sum_{\Im z=0} \operatorname{Res} f(z).$$

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References

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